1. PROBLEMS OF STRENGTH OF MATERIALS

1.1. The Object and the Problems of Strength of Materials – S1

Strength of Materials is a subject of **general technical knowledge**, located between the physical-mathematical sciences and the specialized subjects of engineering. It is a logical follow-up of the theoretical mechanics, a development thereof by introducing **the mechanical and elastic properties of the materials** into the calculations.

Strength of materials aims to establish the methods and the procedures for calculating the stresses, strains and deformations that occur at different points on the mechanical members, when subjected to forces, as well as to establish and use the relations between the stresses and the section dimensions.

Solving the problems of strength of materials takes into account the following three aspects:

I. **the static aspect** which establishes, based on the laws of mechanics, the relations between the external forces and the stresses (internal forces) and respectively the relations between stresses and strains;

II. **the geometrical aspect**, by which the deformations of a body under the action of the forces are analyzed;

III. **the physical aspect**, by means of which the connection relations (laws) between forces and deformations, as well as the mechanical-elastic properties of the respective material are experimentally determined.

Strength of materials solves the following **three categories** of problems:

a) **verification problems**, by means of which it is determined whether a certain dimensions mechanical member subjected to forces meets the strength, rigidity and stability conditions;

b) **problems of calculating the maximum applicable load,** by which, knowing the material and its mechanical and elastic properties, the dimensions and the stresses acting on the element, one can determine the value of the loads that it can withstand;

c) **dimensioning problems,** through which the optimal dimensions of the designed parts are established.

Each of these problems is solved by a **calculation of strength**. This is based on **two criteria**:

I. **the good functioning criterion**, which means ensuring the designed part in terms of:

a) - strength;

b) - rigidity;

c) - stability.

II. **the efficiency criterion**, which makes sure that the designed part represents the most economical possible solution in terms of material and labor consumption.

A calculation of strength is considered appropriate when it meets the two criteria simultaneously.

The good functioning criterion implies meeting the following conditions: - **S2**

a) Each mechanical member of an assembly **must withstand** all the loads that act on it during its entire operating life and therefore **the strength condition is required first**. For this purpose, the Strength of materials teaches how **to choose the appropriate material**, **the shape of the most advantageous section** and **to establish the relations between the cross-section and the loads**, so that at the maximum loads, the stresses that occur in the section of the respective mechanical member to be lower than those causing the fracture.

b) **The rigidity condition** sets the **limit values of the deformations** of the mechanical member of an assembly during the maximum load in operation. Therefore, the *Strength of materials* establishes the relations between the cross-section of the body and the deformations that occur due to the action of the forces and they are used for the calculation of strength (verification, load capacity calculation and dimensioning). The ability of the bodies to have small deformations when subjected to forces is called **rigidity**.

c) **The stability condition** requires maintaining the initial equilibrium shape of the mechanical member under the action of forces. Often in practice there are cases when the dimensions of the mechanical member meet the conditions of strength and rigidity set for the maximum load, but, at lower forces, they lose the stability of the initial equilibrium shape. The phenomenon occurs by the sudden appearance of a very large deformation that can often lead to the fracture of the respective mechanical member and the destruction of the entire construction.

The classic example of the loss of stability of the equilibrium shape is the case of a long straight thin (slim) compressed beam. Subjected to small forces the beam retains its rectilinear shape. If the force increases up to a certain value, the beam suddenly bends, even breaks. The phenomenon is known as **compression buckling** or **loss of stability**, and the force at which the phenomenon occurred is called **critical buckling force**.

1.2. Terminology

Strength of Materials uses specific notions of other disciplines such as mathematics, physics, mechanics, materials technology, etc., but also symbols and notions of its own.

1.3. Classification of Bodies in Strength of Materials – S3

Of all the properties of the mechanical members, in Strength of materials, only those characteristics necessary for the calculation of strength are maintained, ignoring the others. For this purpose the bodies are schematized in **mathematical models** that have certain **mechanical and elastic properties**. As a result, the bodies will fit into the following five models: **wire, beam, membrane, plate** and **solid**. Through these models, Strength of materials schematizes, through a **method of calculation**, the numerous machinery parts and construction elements and, therefore, the calculation of strength is widely applied.

According to their geometry, the bodies are divided into three groups:

a) **Medium fiber bodies**, those with one size, namely length, much larger than the other two, width and thickness. They are defined by:

- **the longitudinal axis** - which can be straight, curved, broken line, etc.
- **the cross section** - which can be constant or variable along the longitudinal axis.

This category consists of:

- **wires** - which can be subjected only to tensile stress and practically show no strength to shear or compressive stress;
- **beams** - which resist to both axial and cross section stresses.

According to their destination and stress, the beams have different specific names: **tensional bar** - when they are tensioned, **poles** - when they are compressed, **beams** - when they are buckled, **shafts** – especially when they are subjected to torsion stress.

A **medium fiber** or **axis** is the geometric place of the centers of gravity of the normal plane sections, on the axis of the beam (or wire), and a **normal section** is the plane section perpendicular to the axis.

b) **The median surface bodies** have one of the dimensions, namely thickness, relatively small in relation to the other two - width and length. This group includes membranes and plates.

- **The membranes**, which are very thin, do not withstand shear or compressive stresses, only tensile stress.
- **The plates**, flat or curved, can also overtake shear and compressive stresses. Examples of plates are lids and walls of tanks, domes, floors, etc. and of membranes are tent cloth, damping membranes, etc.

c) **Solids** or **massive bodies**, have dimensions which fit in the same size category. Examples: bearing balls and rollers, foundation blocks, etc.

The calculations of strength differ from one group to another, being very simple for wires and straight beams, increasing in complexity for curved beams and frames, and becoming particularly complicated for plates and blocks.

Strength of materials shows the way of determining the stresses, strains and deformations in the simplest and most frequently used bodies and for this reason the fundamental and widely discussed topic of the course is the study of the straight beam, with a constant or variable cross section.

The model of a straight beam (fig. 1.1,a) is schematized in fig. 1.1 b. Thus, the model of the beam contains **the axis** of the beam of length **L** drawn with a **thick line** in the figure and **the cross-section**, rectangular in this case, of width b and height h.

The axis system attached to the model is a **straight triple orthogonal system** with **Ox- the axis of the beam** and the **yOz** system, the main central axes of the section.

As a whole, all these models can be called **mechanical members**. Further on, the MM symbol will be used for the general notion of mechanical member in the singular form and (MM) for the plural form.

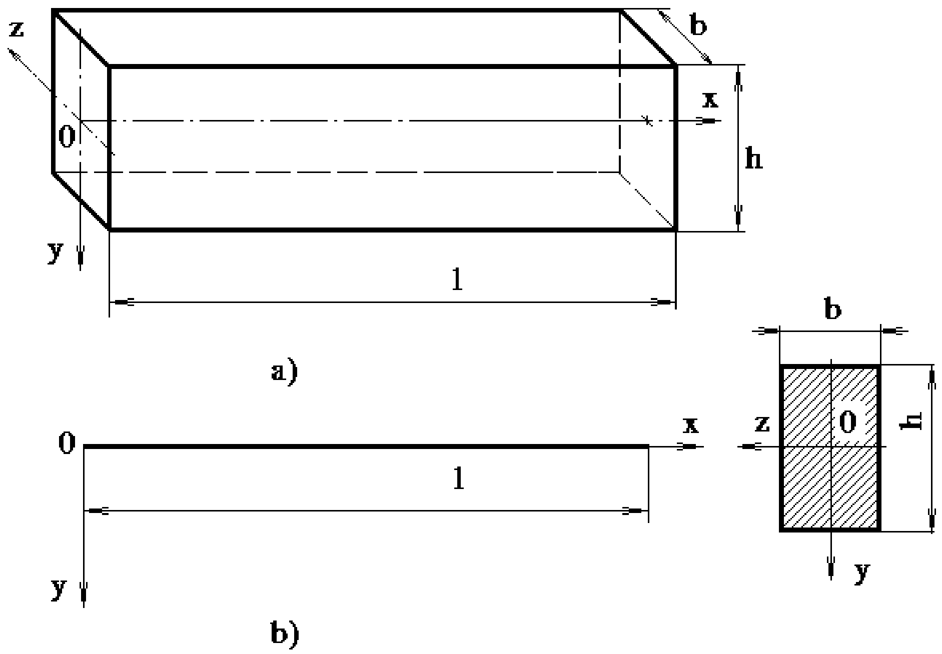


Fig. 1.1.

1.4. Basic Hypotheses of Strength of Materials - S4

To be able to establish the simple calculation relations, in Strength of materials certain hypotheses are used referring, both to the structure of the materials and to their behavior under the action of the applied loads. These hypotheses are sometimes consistent with reality, and, at other times, they represent simplifications of real phenomena, which lead to experimentally verified and therefore acceptable results, intended for the strength of materials. A primary hypothesis was the schematization of the bodies in wires, beams, membranes, plates and solids.

The basic hypotheses of the strength of materials are the following:

I. The hypothesis of the continuous environment, by which it is admitted that the **MM material is considered to be a continuous environment that occupies the entire space delimited by its volume**. This assumption corresponds satisfactorily to the amorphous materials, but does not correspond to the reality of the crystalline ones. The assumption is necessary because the dimensions in the strength of materials, such as stresses, displacements, strains, etc. can be written as locally continuous functions and not as meshing functions specific to each crystal or particle, allowing the use of calculation and of mathematical analysis methods.

II. **The hypothesis of the homogeneous environment**, which admits that **the MM material has the same physical dimensions at all points in its volume**. Nor does this assumption fully agree with reality, especially in the case of concrete, wood or even metals. Thus, hard crusts and mechanical properties different from that of the core are created in metals through various thermal or mechanical treatments.

III. The Hypothesis of the Isotropy. Materials are considered to be isotropic when they have the same elastic properties E, G and ν in all directions. Otherwise, the materials are considered to be **anisotropic**. In Strength of materials, all materials are considered isotropic.

IV. The Hypothesis of the perfect elasticity. If the stresses do not exceed **certain limit values**, **the materials used by the engineers are considered to be perfectly elastic**. This means that the deformations produced by the loads are canceled with the cancellation of the loads.

V. The hypothesis of the proportionality between stresses and strains. Related to the field of elasticity stresses, it is considered that there is a linear relation between stresses and strains, i.e. Hooke's law is valid.

VI. The hypothesis of small displacement. With some exceptions, in Strength of materials it is considered that **the deformations of the MM are very small compared to its dimensions.** The assumption is very important **because the static equilibrium equations can be written by relating the forces to the initial non-deformed state of the MM**. Also based on this assumption, **in the analytical calculations, the terms containing strains or displacements at higher powers can be neglected in relation to the terms at the first power** (the first-order theory).

VII. The hypothesis of the proportionality between strains and displacements. In the elastic field it is considered that **there is a linear relation between strains and displacements**. This assumption is a consequence of the assumption of small deformations.

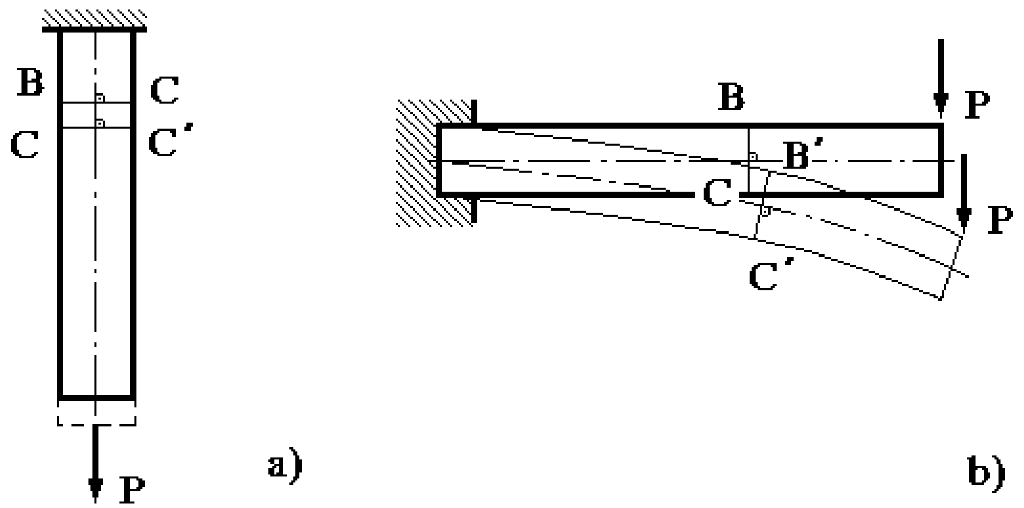


Fig. 1.2

VIII. The hypothesis of plane cross-sections (Bernoulli). The plane and normal sections on the axis of the beam remain planar and normal even after the deformation caused by loads. This assumption is experimentally verified on the outline of the beams and it is also validated inside them.

Thus, in the case of the beam in figure 1.2-a, subjected to tensile stress, the section BC is displaced to B'C' but remains flat and normal on the axis of the beam. The same happens with the beam in figure 1.2-b subjected to bending stress, where the section BC displaces and rotates in position B'C', but remains flat and normal on the axis of the beam.

IX. Saint-Venant's principle. If the forces **acting on a small portion of the MM are replaced with another force system that is statically equivalent to the first, the new distribution of forces produces considerable differences on the place of application as compared to the first but have no effects or an insignificant effect at great distances from the place of force application.**

X. The Principle of overlapping effects. By applying a load **on an MM up to the prescribed limit of the proportionality of the material, the stresses, strains, deformations and displacements that occur in the MM depend exclusively on the size of that load and are not influenced by the effects of**

other previously or simultaneously applied loads. This principle is a consequence of Hooke's law (strains are proportional to loads) and the assumption of the small deformations indicating the first-order theory.

1.5. Safety in Operation. Safety Coefficients. Allowable Strength. – S5

In solving the problems of strength of materials, certain conditions may be imposed on the dimensioned or verified (MM), which will ensure their good functioning throughout their operating life. These conditions are:

a) strength conditions;

b) rigidity conditions;

c) stability conditions.

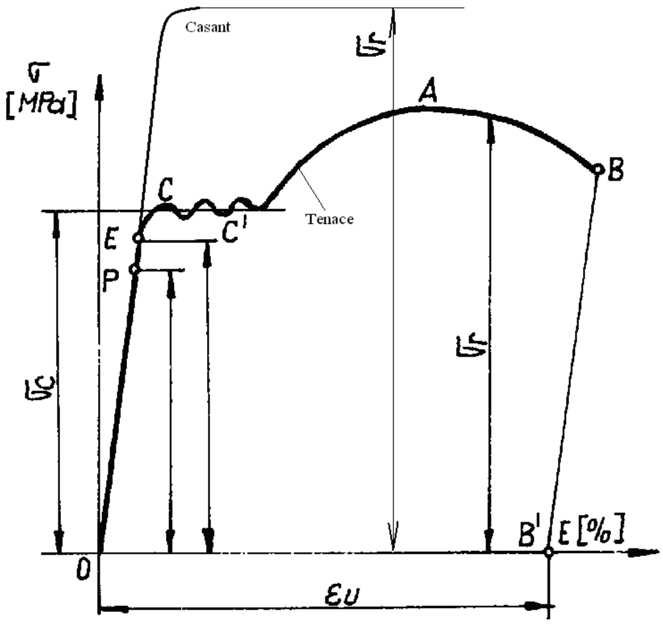
1.5.1. Strength conditions

We consider that an MM is appropriate in terms of its strength conditions, when the stresses that occur in it due to the loads do not exceed certain limits, conventionally established but correlated with the mechanical properties of the material from which the MM is made.

The conventional value chosen in calculation, from practice, for the maximum stress that can be produced in a part, under given material and load conditions is called allowable strength.

Depending on the deformations that occur, up to fracture, the materials are divided into two groups:

- **ductile**, which deform a lot before fracturing (e.g. low and medium strength steels);
- **brittle,** which do not deform or deform very little, without rupture constriction before fracturing (example: cast iron, glass, high strength steel, etc.).



The allowable strength can be defined compared to a **limit state as hazardous** and must be avoided.

In the case of **ductile** materials, which have the flow limit σ_c, the allowable strength is defined by the relation:

$$\boldsymbol{\sigma_a = \frac{\sigma_c}{c_c}} \qquad \textbf{(1.1a)}$$

where: c_c is the **safety coefficient** in relation to the flow limit.

By choosing a correct safety coefficient in the calculations, reaching the flow limit will be avoided, therefore large deformations, which can put the part out of operation, will not occur.

For **brittle** materials, the allowable strength is defined according to the fracture strength σ_r:

$$\boldsymbol{\sigma_a = \frac{\sigma_r}{c_r}} \qquad \textbf{(1.1b)}$$

where: c_r is **the safety coefficient** in relation to the fracture strength.

The tests conducted on different (**MM**) showed which should be the most suitable values of the safety coefficients and therefore of the allowable strengths. For example, if we refer to steel, the allowable strength must be lower than the flow limit but also than the elasticity and proportionality limits.

When **choosing the safety coefficient c** we have to take into account the following factors:

a) the nature of the material and the manufacturing technology. Each material has certain mechanical properties that determine the allowable strength $\boldsymbol{\sigma}_a$. The more non-homogenous the material is, the higher the safety coefficient. Thus, the safety coefficient for cast iron is higher than for steel, and the safety coefficient for concrete or wood is higher than for metals. The uneven structure of the material, the existence of molding, forging or rolling crusts are technological factors that have a negative effect on the allowable strength and therefore a higher safety coefficient will be considered**.**

b) The type of load. The performed mechanical tests (tension, compression, bending, etc.) have proven that the materials have different mechanical properties depending on how the stress is applied. However, some materials have equal allowable strengths for different stresses, for example, steel for tension, compression, bending.

c) The action of stresses over time. When a MM is stressed with static loads, the safety coefficient is lower than for time-varying loads or for shock loadings. It has been experimentally observed that a material with a fracture strength σ_r, subjected to time-varying stresses over time fractures at values σ_{max} lower than σ_r. This phenomenon was given the name of **material fatigue**. The upper limit value of σ_{max}, where the material endures a very large number of cycles (e.g. $5 \cdot 10^7$ to 10^8 cycles) without fracturing, is called **fatigue strength**.

d) The way of evaluating the loads and applying the calculation assumptions. Less precisely the loads are evaluated, the more the assumptions and the calculation schemes are approximated, the lower the allowable strengths and the higher the safety coefficients**.**

e) The operating life of the part. For parts with short operating life, lower safety coefficients, thus higher allowable strengths, can be calculated.

f) Temperature. High or low temperatures adversely affect the allowable strengths. For important (MM) that will operate at high or low temperatures, the allowable strength is chosen according to the mechanical properties at that temperature.

1.5.2. Rigidity conditions

The operation of parts is possible only when their deformations do not exceed certain limits. For example, a shaft with large bending deformations causes premature bearing wear. For this reason, when calculating strength, certain limits are imposed for the size of the deformations and it is considered that the MM must meet certain rigidity conditions.

1.5.3. Stability Conditions

Related to the problems of elastic stability, although the strength conditions are met, at certain values of the loads, called **critical values**, the parts can lose their standing balance, which leads to their destruction. These (MM) must meet the **stability conditions**, that is, the loads applied must be lower than the critical ones.

Some indicative values of the allowable strengths are given in Annex 1. This table shows that the admissible bending strengths are usually 10-20% higher than the tensile strengths, while the shear and torsion strengths are 60-80% of the tensile ones. An exception to this rule is cast iron, which has allowable compressive strengths 2 to 5 times greater than the tensile strength.

2. EXTERNAL FORCES AND INTERNAL FORCES

2.1. External Forces. Classification. - S6

The engineering constructions are made of one or more (MM). In Strength of materials, each MM or subassembly is analyzed **only in the state of equilibrium under the action of the external forces**.

In Strength of materials the notion of **external force** includes both the **forces applied** on the MM surface and those **distributed throughout the mass of the material** such as: weight, inertial loads, electromagnetic forces, forces due to arrested expansion, etc., as well as the **bonding forces between (MM)** called **reactions**.

External forces can be classified as follows:

a) according to their **nature**:

- **active loads** or **forces**;
- **reactions** or **bonding forces**.

b) according to the **place of application**:

- **surface** or **border forces**, applied outside the MM;
- volume or mass forces, distributed throughout the volume of the MM.

c) according to the **size of the surface** on which they are applied, the surface forces may be:

- **concentrated**, which are applied punctually and are a schematization of the forces distributed over a very small surface, in relation to the surface of the (MM);
- **distributed**, which are **uniformly** or **variably** distributed on a surface or along a line.

The concentrated forces are measured in N, kN, MN, etc, while those distributed on the surface are measured in N/m^2 or Pa, N/mm^2 or MPa, kN/m^2, etc. and those distributed along a line in N/m, kN/m, etc.

The forces applied on (MM) can be classified as follows:

a) According to their **origin:**

- permanent loads, which maintain their intensity constant (example: the dead load of the MM);

- live loads consisting inv those resulting from the functional role of the MM (examples: the weight of vehicles for a bridge, the payload for the means of transportation, the cutting force for tools, etc.);

- accessory loads that occur during operation (examples: inertia forces, frictional forces, arrested expansion, etc.);

- accidental loads, acting intermittently and irregularly (examples: the action of the wind, the weight of the snow, etc.);

- extraordinary loads, which act accidentally, but can have a catastrophic effect (examples: fires, explosions, floods, earthquakes, etc).

The permanent, live and accessory loads are called **fundamental loads**.

b) According **to the behavior over time**, they can be classified into:

- **static loads**, which apply slowly, and then maintain their intensity constant**;**

- dynamic loads, which are applied with relatively high variable speed and which can be:

- suddenly applied loads, producing shocks;

- time-varying loads whose intensity varies periodically according to a certain law**.**

c) According to **the position of the load on MM:**

- static load, acting in the same place for the entire operating life of the construction (example: the dead load);

- dynamic load, whose position is variable (example: the weight of a vehicle on a bridge).

2.2. Reactions – S7

Reactions or **bonding forces** are the mechanical action of the connections between the MM and other (MM) and occur under the action of loads on the MM.

The bonds cancel one or more degrees of freedom of the MM, restricting its movements. According to **the axiom of connections**, **the effect of bonding an MM, subjected to loads, can always be replaced by appropriate reactions (bonding forces), which are determined by the equilibrium conditions**. When the number of distinct equilibrium equations equals that of the MM reactions, we have a **determined static system**, and when the number of the equilibrium equations is smaller than the number of reactions, we consider the system to be **statically indeterminate**. **The degree of indeterminacy** is given by the difference between the number of reactions and the number of equilibrium equations. The indeterminacy is solved in Strength of materials by introducing the **geometric deformation conditions.**

Unlike theoretical mechanics, in Strength of materials, **the forces are vectors connected** to the point of application. **Changing the point of application of the force does not change the equilibrium, but it can change the stresses occurring in the MM**.

2.3. Internal Forces – S8

Internal forces or **stresses** occur within the MM when it is loaded by external forces. In order to determine the stresses, Strength of materials uses **Cauchy's method of sections**. This method is equivalent to the general equilibrium theorem: **if an MM is in equilibrium under the action of a system of forces, then any part of this body is also in equilibrium under the action of the forces corresponding to that part**.

This method consists of:

- the imaginary sectioning of the MM, where the corresponding internal forces (stresses) are to be determined;

- the representation, on the obtained MM sections, of the external and internal forces;

- writing the equilibrium equations for the external loads and the stresses represented for one of the parts of the sectioned MM.

We consider an arbitrary beam subjected to a system of forces F_1, F_2...F_n (figure 2.1-a), and sectioned by **an imaginary plane Q, normal on the axis of the beam.** Two parts are obtained through sectioning: ① and ②. The two sides of the beam are balanced by the internal distributed forces p, which occur on the section side A (fig. 2.1, b). The forces distributed on the surface A of part ② are reduced in the center of gravity O_2 to a resultant force R_2 and a resultant moment M_{02}. At the same time, these are the effect of part ① on part ②. Therefore, the forces p on the side A of part ② are equivalent **to the reduction torsor in 0_2** of the forces acting on part ① (fig.2.1c).

Similarly, if part ① is represented, the action of part ② on part ① is equivalent in O_1, with the resultant $R_1$and the resultant moment M_{01}.

The influence of part ① on part ② is equal and contrary to the influence of part ② on part ① (according to the action-reaction principle) which leads to:

$$\overline{\mathbf{R}_1} = \overline{\mathbf{R}_2} = \overline{\mathbf{R}} \qquad \text{and} \qquad \overline{\mathbf{M}_{01}} = \overline{\mathbf{M}_{02}} = \overline{\mathbf{M}_0}\,.$$

The elements of the reduction torsor in the center of gravity of the section of the forces acting on the left side are equal and in the opposite direction with the elements of the reduction torsor, in the same point, of the forces acting on

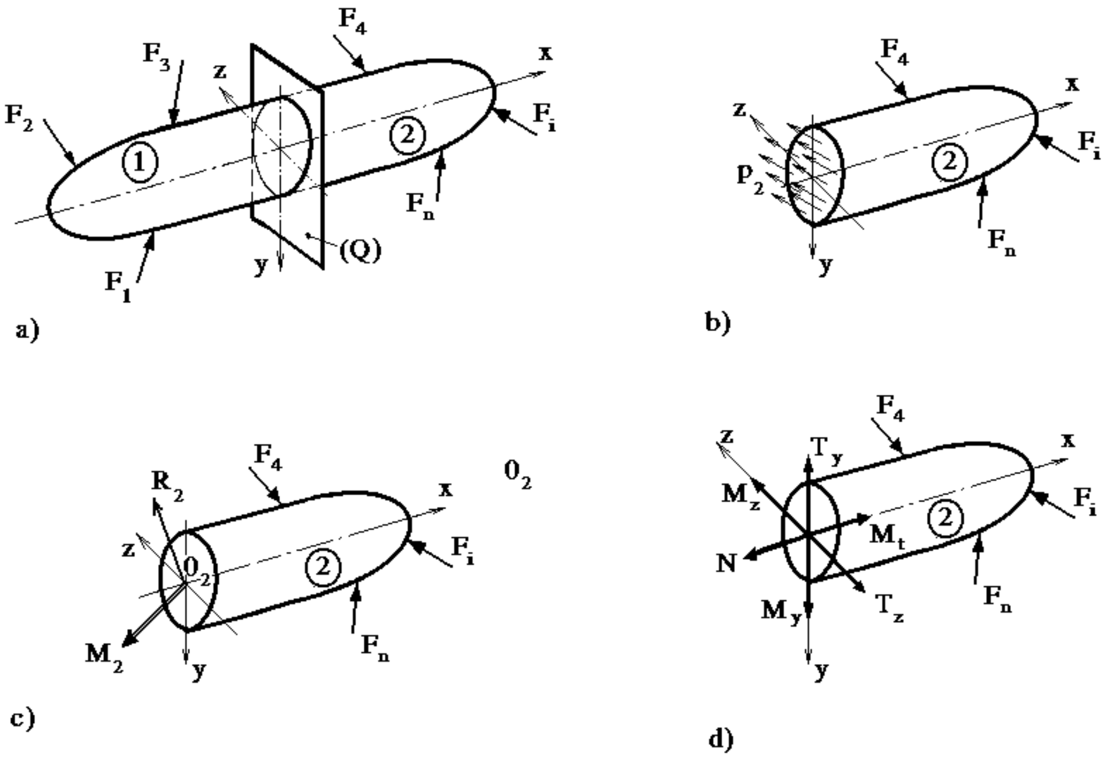


Fig. 2.1

the right side.

The elements R_1, M_{01}, and respectively R_2, M_{02}, that ensure the equilibrium of each part are called **internal forces.**

They are, at the same time, the resultant and respectively the resultant moment of the elementary internal forces that occur between the particles of the two parts under the action of the loads. By separating the two parts through an imaginary plane, the internal forces have been transferred into the category of external forces and considered as such.

By designing the elements of the reduction torsor in O, on the coordinate axes, six components are obtained: three forces: N, T_y, T_z and three moments: M_t, M_y, M_z (fig.2.1, d). **The components N, T_y, T_z, M_t, M_y, M_z** are called sectional stresses or section stresses and will be referred to as **STRESSES**. Each stress has a **name**, a corresponding **displacement** (deformation) and it **causes a simple load** on the beam.

The normal force or the **axial force N** (fig. 2.1, d), is equal to the algebraic sum, taken with changed sign, of the projections on the x-axis of all the forces on the left (or right, taken with the same sign) of the considered section:

$$\mathbf{N = -\sum_1 F_x = \sum_2 F_x} \ . \qquad \mathbf{(2.1)}$$

where 1 means that the forces on the left are considered, and 2, that the ones on the right are considered.

The normal force is considered to be **positive** when it causes a tensile stress, **which elongates the beam** and **negative** when it causes **a compression stress**, which **shortens the beam.**

The shearing force T_y, respectively T_z, is equal to the sum of the projections on the 0y and 0z axes, in the plane of the section, taken with a changed sign, of all the forces located to the left (or to the right with the same sign) of the considered section:

$$\mathbf{T_y = -\sum_1 F_y = \sum_2 F_y; \qquad T_z = -\sum_1 F_z = \sum_2 F_z} \ .$$

(2.2)

The shearing force T_y is **positive** if **the section displaces in the opposite direction of the 0y axis**, in plane **x0y**, and T_z, in the opposite direction of the 0z axis. **The shearing forces cause shear or shearing stress.**

The bending moment M_z, and also **M_y,** is equal to the sum of the moments in relation to the 0z axis and the 0y axis of the plane of the section, of all the torques of forces and the moments of the forces, located to the left (or to the right taken with minus) of the considered section:

$$\mathbf{M_z = \sum_1 M_z = -\sum_2 M_z} \ \text{ and } \ \mathbf{M_y = \sum_1 M_y = -\sum_2 M_y} \ . \qquad \mathbf{(2.3)}$$

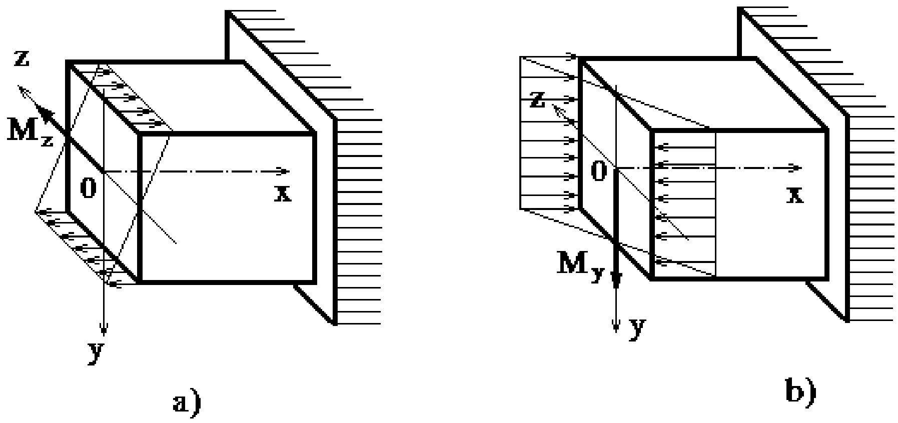


Fig. 2.2

The bending moments cause the bending stress. The deformation caused by the bending moment is the twisting of the section around the respective axis: M_z, around the Oz axis and M_y around the Oy axis. Moment M_z is considered to be **positive**, when **it compresses the upper fiber and elongates the lower one**, and M_y is positive when it compresses the fiber on the positive side of the Oz axis and elongates the fiber from the negative side (fig. 2.2).

The twisting moment M_t is equal to the algebraic sum of the moments of the forces and the torques located to the left of the section (or to the right taken with a minus sign) relative to the Ox axis:

$$\mathbf{M_t = \sum_1 M_x = -\sum_2 M_x} \,. \tag{2.4}$$

The torque is positive when the forces or the force couples on the left of the section rotate clockwise and those on the right counterclockwise.

The simultaneous presence in the beam section of two or more stresses causes a compound stress in the beam.

In general, the stresses on the right side of the section are determined (O_2yz in fig.2.3,d) and in this case the forces on the left side of the section decrease. When it is easier to decrease the forces on the right side, then the stresses on the left side are obtained, which have opposite signs than those determined in the first case. If the forces on the left side of the section have been deduced and they must be related to the right side then their sign will be changed.

It should be noted that the representation of the interaction, by forces applied in O, is a simple conventional representation of the complex phenomenon of interaction between the two parts, (fig.2.1, b).

Note: It is easier to obtain the section stresses by doing the following:

a) we establish in which part of the section there are fewer forces and **only the forces in that part (left or right) are taken into consideration**;

b) each force in that part is decomposed according to the directions of the axes in the section;

c) each component obtained from the forces is reduced in the center of gravity of the section;

d) the projections of the forces and of the moments corresponding to each axis are summed, taking into account the rule of signs, and the following is obtained:

- $\mathbf{N}$ = the sum of the projections of forces on the Ox axis;
- $\mathbf{T_y}$ = the sum of the projections of forces on the Oy axis;
- $\mathbf{T_z}$ = the sum of the projections of forces on the Oz axis;
- $\mathbf{M_y}$ = the sum of the projections of the moments on the Oy axis;
- $\mathbf{M_z}$ = the sum of the projections of the moments on the Oz axis;
- $\mathbf{M_t}$ = sum of the projections of the moments on the Ox axis.

2.4. Stress Functions

The stress values in the section (N, T_y, T_z, M_y, M_z, M_x) vary along the beam, depending on the way they are applied and on the shape of the beam. One of the main problems when calculating strength is to know the values of the stresses of each cross section. Thus, the variation of each stress is expressed according to the coordinates of the axis points and **a stress function** is obtained. For a straight beam, which has the axis oriented on Ox, the stress functions are expressed depending on the x-coordinate of the section: **$N = N(x)$; $T_y = T_y(x)$;... $M_z = M_z(x)$.**

The variation of stresses along the axis of the beam, under the action of the static loads, can best be traced on **the stress diagrams**. These are graphical representations of the stress functions according to the coordinate of the "x" section on the axis of the beam. **The stress diagram is obtained by drawing a thin line that connects the points that meet the requirements of the equation of that stress function**. This is **represented along a reference line, drawn with a thick line, parallel and equal in length to the axis of the beam**. Thus, a diagram is drawn for each stress.

In practice, straight or plane curved beams are frequently encountered, which are loaded with forces contained in the beam's longitudinal plane of symmetry. Figure (2.3,a) shows such a beam and the plane of the forces is marked xOy. The reactions and the stress have been determined in the section on the "x" coordinate of bearing 1. Figure (2.3, b) contains the respective beam on which the reactions and the internal stresses in the "x" coordinate section were represented.

In this particular case the following stresses can be determined:

a) **the axial force,** equal to the algebraic sum of the projections of the external forces applied on the left (or right) of the considered section on the axis of the beam;

b) the shearing force, $T = Ty$, equal to the algebraic sum of the projections on the Oy axis of all the forces on the left (or right) of the considered section;

c) the bending moment, $M = M_z$, equal to the algebraic sum of the moments of the forces in relation to the axis of Oz, of all forces and moments located on the left (or right) of the considered section.

Usually, for drawing the stress diagrams for loads contained in a single plane, the plane design in figure (2.3,d) is used. The sectional stresses, on the left and on the right of the section are represented as in figure 2.3,d.

The sign rule for a plane stress is given in figure 2.4:

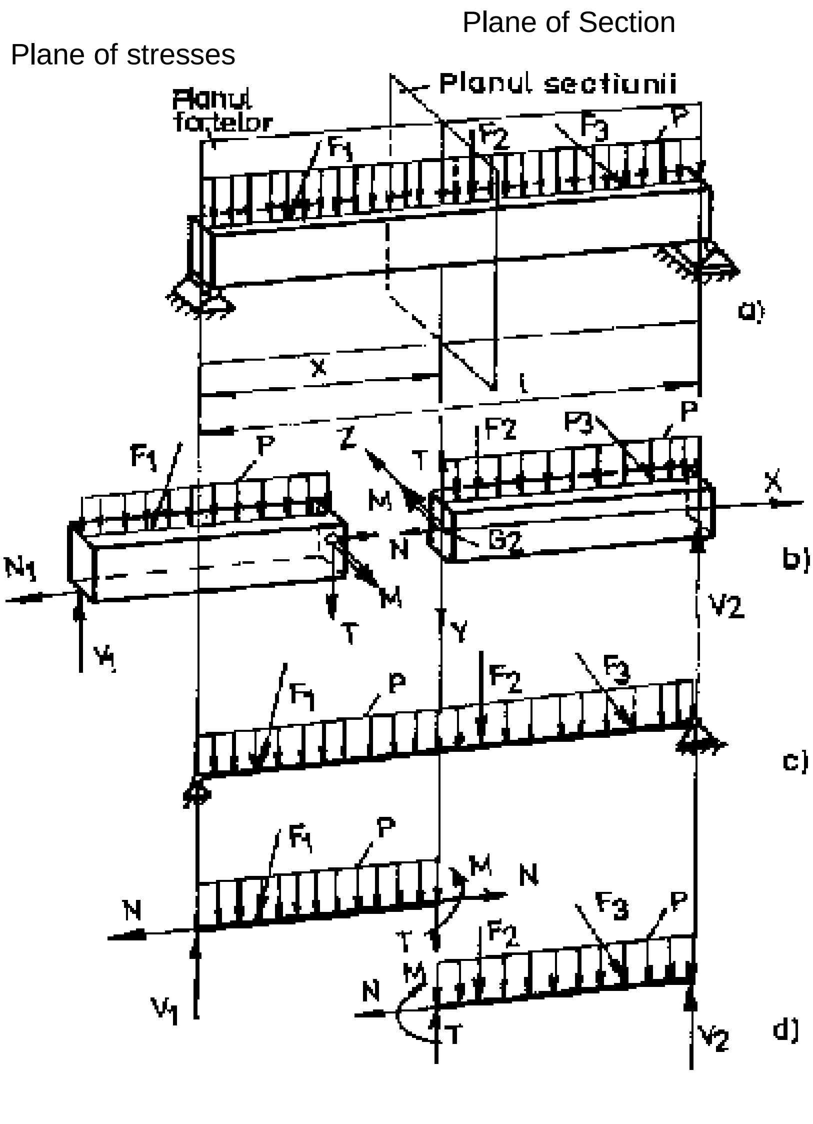


Fig. 2.3

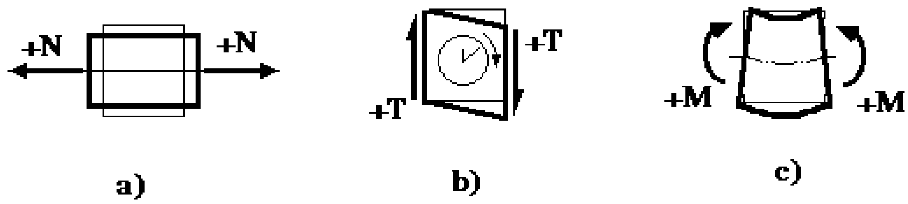


Fig. 2.4

- the axial force N, is positive when it elongates the beam section (fig.2.6,a) and negative when it **shortens** it.

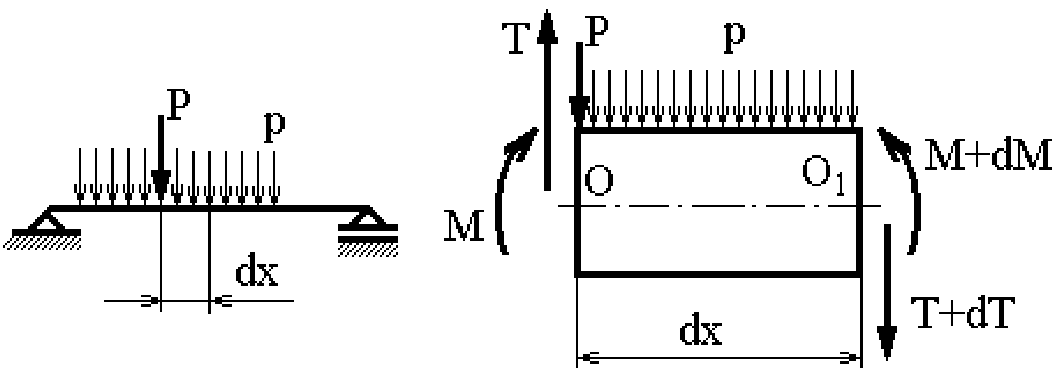
- the shearing force T is positive when it tends to twist beam section clockwise (fig.2.4,b);

- the bending moment M is considered positive when it twists the two lateral sides, bending the fibers, so that the upper fibers shorten and the lower ones elongate (fig. 2.4, c).

2.5. Differential Relations between Loads and Stresses – S9

Drawing the stress diagrams can be much easier if both **the stress functions** and the **differential relations** between the stresses and different loads are known.

In order to establish the differential relations between the loads and the stresses, a straight plane beam subjected to a system of loads contained in the plane of the beam axis is taken into consideration. The beam section is considered to be of infinitely small length dx, (fig.2.5,a).



a) Fig. 2.5 b)

The following loads act upon element *dx*:

- **p**, uniformly distributed along the dx length of the element;
- **T** and **M** concentrated and acting in the section passing through point 0.

Also in figure (2.5,b) stresses: **T, M** were represented in section O and respectively **T+ΔT** and **M+ΔM** in section A. According to (Cauchy's) method of sections, if the initial element is in equilibrium, then a portion of element of length dx will have to be in equilibrium. In this case, the following equations can be written:

$$\sum Y = 0; \quad (T + \Delta T) \cdot -T + P + pdx = 0 \tag{2.5}$$

$$\sum M_O = 0; \quad (M + \Delta M) - M - (T + \Delta T) \cdot dx - p \cdot dx \cdot \frac{dx}{2} - M_e = 0$$

If the products of the small infinities are omitted, the relations (2.5) become:

$$\Delta T + P + p \cdot dx = 0; \quad \Delta M - T \cdot r \cdot dx - M_e = 0. \tag{2.6}$$

These relations contain finite and infinitely small quantity terms. If the infinitely small terms are neglected compared to the finite terms, the following equations are obtained:

$$\Delta T = -P, \qquad \Delta M = M_e \tag{2.7}$$

Neglecting the infinitely small terms can be done (and should be done) only in the vicinity of the point loads. From relations (2.7) it follows: **in the vicinity of a point load, at least one stress proves a rise equal to the value of the component of the point load in the respective direction**. For example, in the vicinity of a transverse point force Y, in the shearing forces diagram, there must be a rise equal to the value of component Y, and in the vicinity of a point moment M_e, in the bending moments diagram, there is a rise equal to the value of moment M_e.

In case there are **no point loads (Y = 0 and M_e = 0)** applied on element dx, then the relations (2.7) must contain only the infinitely small terms. Therefore, the variation of the stresses must be infinitely small, so it is considered:

$$\Delta T \to dT, \qquad \Delta M \to dM.$$

Taking the relations in (2.6) into account, the following is obtained:

$$\frac{dT}{dx} = -p, \qquad \frac{dM}{dx} = T. \tag{2.8}$$

These relations lead to the following:

- by deriving the expression of the bending moment in relation to the "x" variable, the expression of the shearing force is obtained;

- by deriving the expression of the shearing force in relation to the "x" variable, the expression of the distributed load p with a negative sign is obtained.

By deriving the first relation again and taking into account the second, the following is obtained:

$$\frac{d^2M}{dx^2} = \frac{dT}{dx} = -p \,. \qquad \textbf{(2.10)}$$

Notes:

a) Relations (2.8), (2.9) and (2.10) are differential relations of the stress functions **N(x)**, **T(x)** and **M(x)**. The stress diagrams represent the integrals of these expressions.

b) Relation (2.10) shows that the equation of the shearing force can be obtained, either by integrating the expression of the load or by deriving the expression of the bending moment.

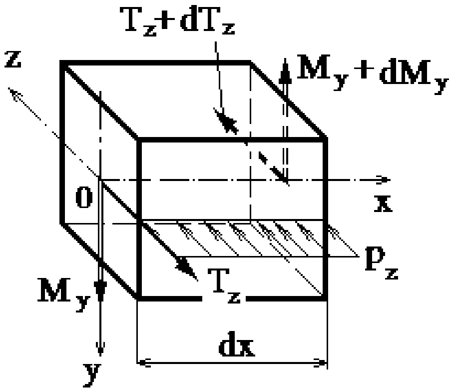


Fig. 2.8

c) If the loads are contained in the xOy plane (fig.2.8), the equilibrium equations are:

$$-T_Z + p_Z \cdot dx + (T_Z + dT_Z) = 0,$$

$$M_Y - T_Z \cdot dx + p_Z \cdot dx \cdot \frac{dx}{2} - (M_Y - dM_Y) = 0$$

Thus, the following is obtained:

$$\frac{dM_y}{dx} = T_z, \qquad \frac{dT_z}{dx} = -p_z\,, \qquad \textbf{(2.11, a)}$$

$$\frac{d^2M_Y}{dx^2} = \frac{T_z}{dx} = -p_z\,. \qquad \textbf{(2.11, b)}$$

2.6. Practical Rules for Drawing the Stress Diagrams – S 10

In the case when the transverse forces are zero **(Y= 0; p= 0)**, relations (2.10) lead to:

$$\mathbf{T = C_1, \qquad M_i = C_1 \cdot x + C_2.} \qquad \mathbf{(2.12)}$$

Therefore, **when the transverse forces are zero, the shearing force is constant and the bending moment varies linearly** (fig. 2.9, a and b). C_1 and C_2 are integration constants and represent the shearing force, respectively the bending moment, at the left or right boundary of the considered section.

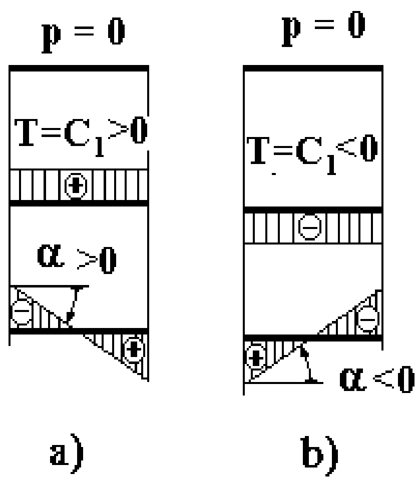


Fig. 2.9

If a uniformly distributed transverse force (p = ct.) is applied on a beam section, then the relations (2.10) lead to:

$\mathbf{T = C_1 - p_1 \cdot x}$ (linear variation),

$\mathbf{M = C_2 + C_2 \cdot x - p \cdot x^2}$ (parabolic variation). **(2.13)**

For this case, several modes of variation of the shearing force and of the bending moment have been represented on a beam section (fig.2.10).

The second relation (2.10) shows that **the shearing force is equal to the slope of the curve of the bending moments**.

Figures 2.9 and 2.10 show that on the section where:

$$\begin{aligned}
&T>0 \;\rightarrow\; M \;\; increases, \\
&T<0 \;\rightarrow\; M \;\; decreases, \\
&T \;\; goesthrough\ zero \rightarrow \;\; M_{\max} \ or \ M_{\min}, \\
&T=0 \;\rightarrow\; M=ct.
\end{aligned} \qquad \textbf{(2.14)}$$

If we consider relations (2.7), when there are point loads present, it results that a sudden change of the slope of the bending moment corresponds to a sudden variation of the shearing force. Hence, **there is a point on the moment diagram where the slope of the tangent changes (it breaks) in the vicinity of the transverse point load**.

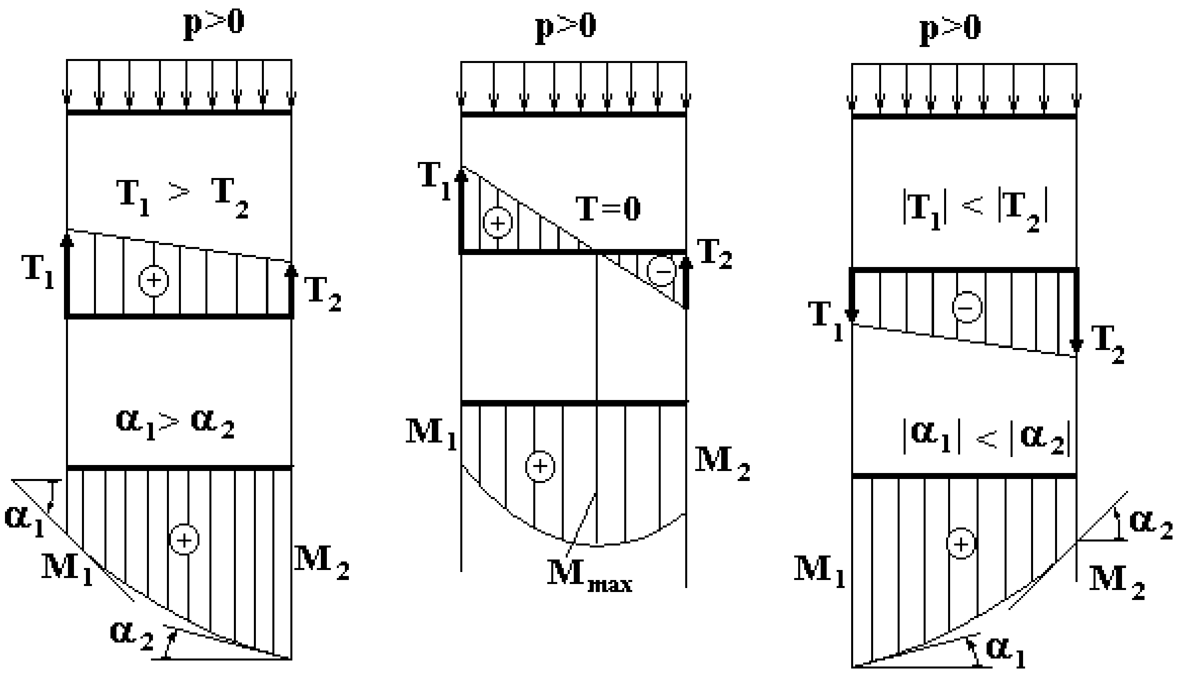


Fig. 2.10.a

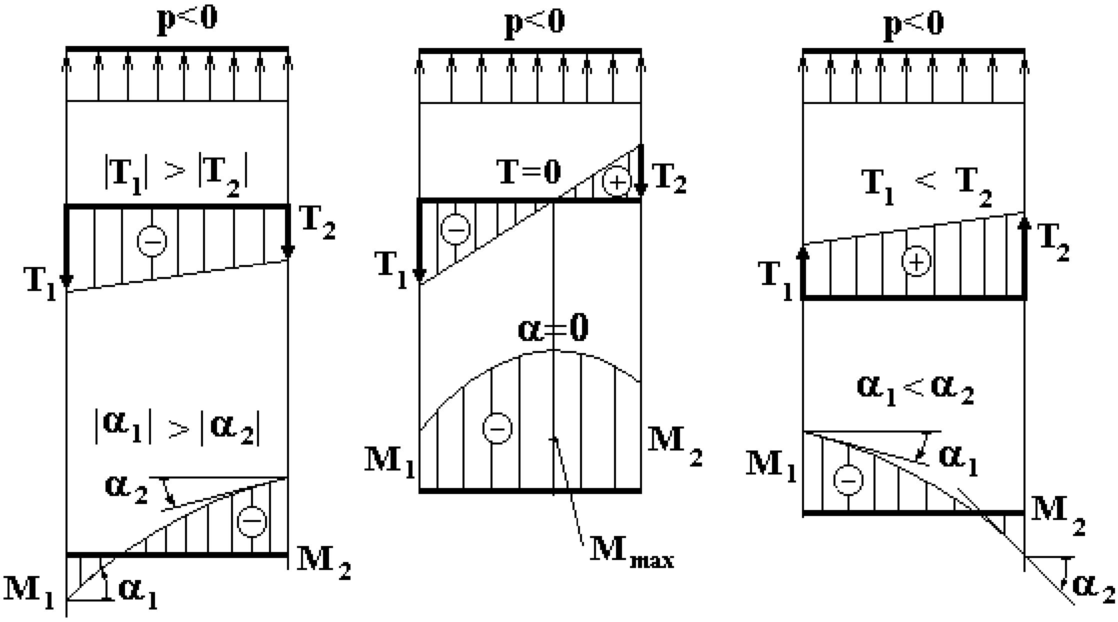


Fig. 2.10.b

In addition to the above mentioned rules, it is necessary to take the following steps when drawing the stress diagrams:

a) the beam is released from any connections, the reactions are represented and their value is determined from the equilibrium equations;

b) a direction of the beam is chosen, that is, an origin of the Ox axis and its direction, which can be from left to right or from right to left, from top to bottom or from bottom to up, etc .;

c) the stress functions are established, i.e. the expressions N(x), T(x) and M(x) for each beam section;

d) for each existing stress, **a thick reference line** is drawn, parallel to the axis of the beam and of the same length therewith;

e) the axial forces, the shearing forces and the positive twisting moments are represented on a scale above the reference line; the positive bending moments are drawn below the reference line;

f) the representation of the stresses in the diagrams is made by drawing **straight segments perpendicular to the reference line, which represent, at scale, the value of the respective stress**.

2.7. Stress Diagrams – 11 a

Stress diagrams are necessary to determine the hazardous section and therefore they are always drawn for every loaded beam. One can immediately notice on the diagram **the loads** and **the most loaded (hazardous) sections**, as well as the limit values of the stresses.

2.7.1. Straight beams acted upon by axial forces

In these cases the external forces acting along the beam are reduced to resultants whose support is precisely the axis of the beam.

Exercise 2.1. Draw the stress diagram for a beam loaded as in Figure 2.11.

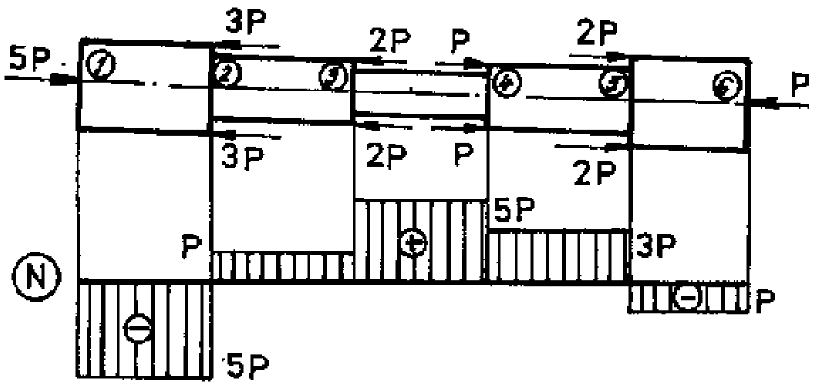


Figure 2.11

The stresses are the following:

$N_{1s}= 0$; $N_{1d}= N_{2s}= -5P$; $N_{2d}= N_{3s}= P$; $N_{3d}= N_{4s}= 5P$; $N_{4d}= N_{5_s}= 3P$; $N_{5_d}= N_{6_s}= -P$; $N_{6_d}= 0$.

Exercise 2.2. A vertical pole loaded with the axial load P = 500 kN consists of two sections and rests on a concrete block. Both the pole with its two sections and the foundation have constant sections and the lengths of figure 2.12. The weight distributed along section 1-2 is q_1= 25 kN/m, on section 2-3 it is q_2= 35 kN/m, and the weight of the foundation is q_3= 90 kN/m. Draw the stress diagrams.

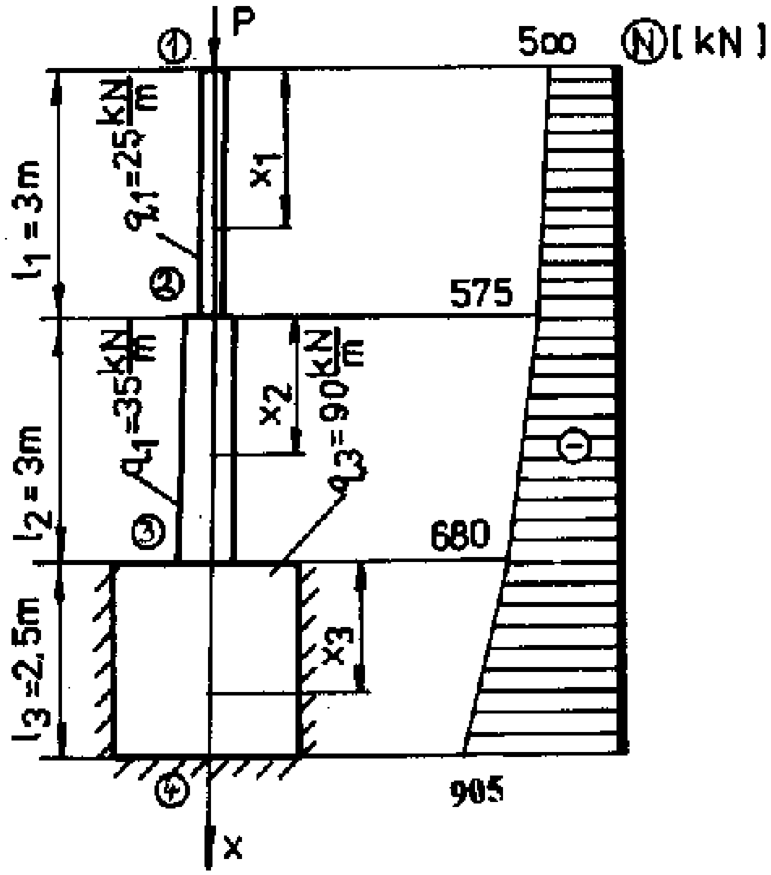


Fig. 2.12

In an arbitrary section, on coordinate x_1, the axial force is:

$N(x_1) = - P - q_1 \cdot x_1$, $N_{x1} = - 500 - 25 \cdot x_1$,

therefore, it linearly varies.

The limit values are:

$N_1 = - 500$ kN, $N_2 = - 500 - 25 \cdot 3 = - 575$ kN.

In an arbitrary section on part 2-3, the axial force has the following expression:

$N(x_2) = - P - q_1 \cdot l_1 - q_2 \cdot x_2$, and the limit values will be:

$N_2 = - 500 - 25 \cdot 3 = - 575$ kN, $N_3 = - 500 - 25 \cdot 3 - 35 \cdot 3 = - 680$ kN

In a section on the foundation, the axial force is given by the expression:

$N(x_3) = - P - q_1 \cdot l_1 - q_2 \cdot l_2 - q_3 \cdot x_3$, and the limit values are:

$N_3 = - 500 - 25 \cdot 3 - 35 \cdot 3 = - 680$ kN, $N_4 = - 500 - 25 \cdot x_3 - 35 \cdot x_3 - 90 \cdot x_2 = - 905$ kN.

The variation diagram of the axial stresses is shown to the right of the beam.

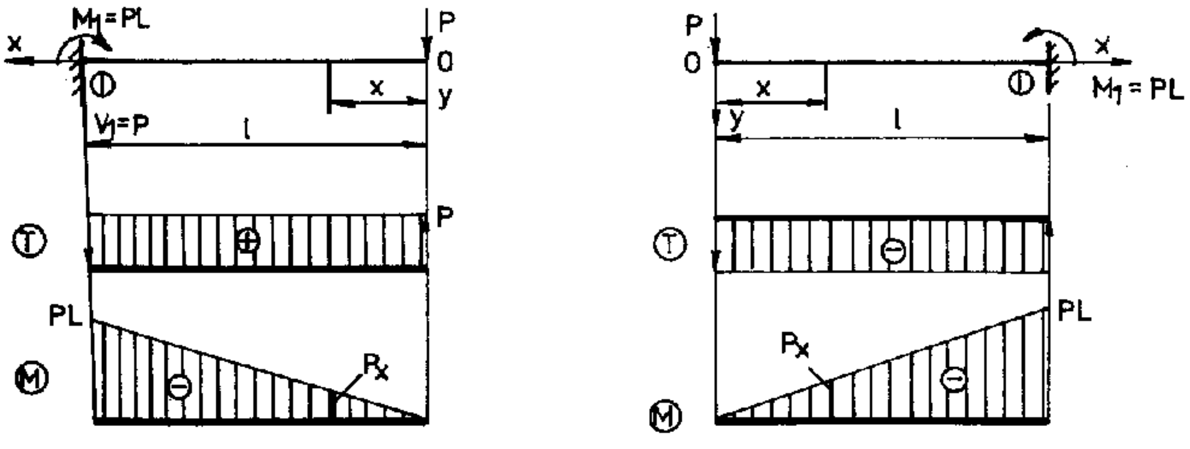
2.7.2. Straight bar (beam) subjected to bending

To begin with, the straight bars subjected to vertical external forces located in one of the bar's longitudinal symmetry planes will be considered. In this case, in the cross-sections of the bar, when subjected to loads, axial forces, shearing forces and bending moments occur.

2.7.2.1. Cantilevers

Regarding the cantilevers (with a fixed end and a free end) the stress diagrams can be drawn even without previously calculating the reactions. In this case, the origin of the reference system is considered in the free end, and the reactions will be equal to the values of the stresses in the fixed end.

Exercise 2.3. Cantilever with a fixed end and subjected to a point load at the other (fig. 2.13). In figure (2.13, a), the cantilever has the free end on the right, while in figure (2.13, b), the free end is on the left.



a) **b)**

Fig. 2.13

The stress functions for the cantilever in figure (2.13,a) are the following:

$T_x = P =$ ct. $M_x = - P\cdot x$ (linearly varies) and has values $M_0 = 0$ and $M_1 = - P\cdot L$.

The stresses acting on the cantilever in figure (2.13,b) are:

$T_x = - P =$ ct. $M_x = - P\cdot x$, $M_0 = 0$ and $M_1 = - P\cdot L$.

Note: The shearing forces have equal absolute values, but different signs.

Exercise 2.4 Cantilever subjected to a uniformly distributed transverse force (fig.2.14).

In section x the stresses are:

$T_x = - p \cdot x$ (straight line),

$M_x = - p \cdot x(x/2) = - p \cdot x^2/2$ (parabola),

and the limit values become:

$T_0 = 0$; $T_1 = - p \cdot L$; $M_0 = 0$;

$M_1 = - p \cdot L^2/2$.

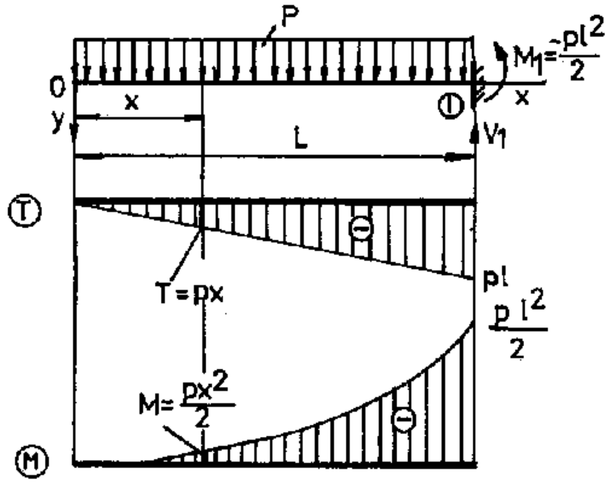


Fig. 2.14

The reactions in the fixed end are:

$V_1 = p \cdot L$; $M_1 = - p \cdot L^2/2$.

Exercise 2.5. Cantilever subjected to a linearly distributed load (fig. 2.15).

The load is determined by the maximum intensity of load p_0. The total load on the bar is $p = p_0 \cdot L/2$ and the load intensity in an arbitrary section, at distance x from the end, is:

$p = p_0 \cdot \left(1 - \frac{x}{L}\right)$. The stresses in section x are:

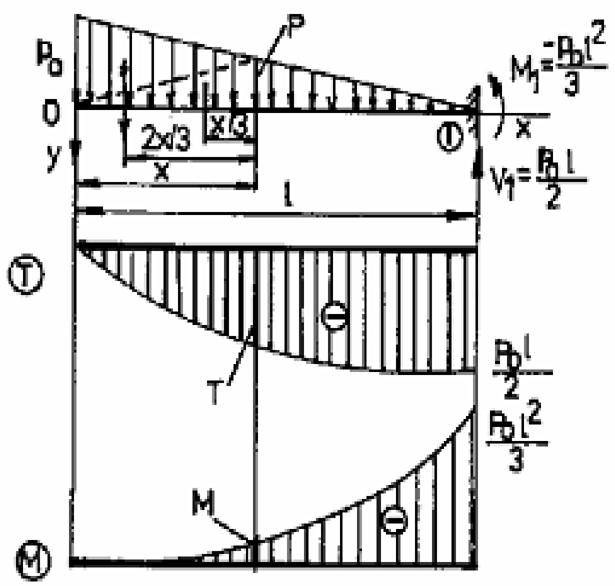


Fig. 2.15

$$T_x = -(p_0 + p) \cdot \frac{x}{2} = -\frac{p_0 \cdot x}{2} \cdot \left(2 - \frac{x}{L}\right),$$

$$M_x = -p_0 \cdot \frac{x}{2} \cdot \frac{2x}{3} - p \cdot \frac{x}{2} \cdot \frac{x}{3} = -\frac{p_0 \cdot x^2}{6} \cdot \left(3 - \frac{x}{L}\right).$$

It is noted that the shearing force varies according to a 2nd degree parabola, and the bending moment according to a 3rd degree parabola. In the two ends of the cantilever the stresses will have the following values:

$T_0=0, \quad M_0=0, \quad T_1= - p_0 \cdot L/2, \quad M_1= - p_0 \cdot L/3,$

and the reactions will be:

$$V_1 = p_0 \cdot \frac{L}{2}, \quad M_1 = -\frac{p_0 \cdot L^2}{3}.$$

Notes:

a) The shearing force in an arbitrary section x is equal to the surface of the diagram of the forces distributed along Ox;

b) The bending moment in a section x is the product between the resultant of the forces along Ox and the distance from section x to the resultant.

2.7.2.2. Simply supported bar (beam)

The simply supported beam has a simple bearing at one end and a joint at the other. In the joint, two components of the reaction will be considered, namely V vertically and H horizontally. Only one reaction occurs in the simple bearing, namely a normal force on the bearing surface.

The distance between the two bearings is L and **is called the span of the beam**.

Exercise 2.6. Simply supported beam subjected to a point force Q acting obliquely (fig. 2.16).

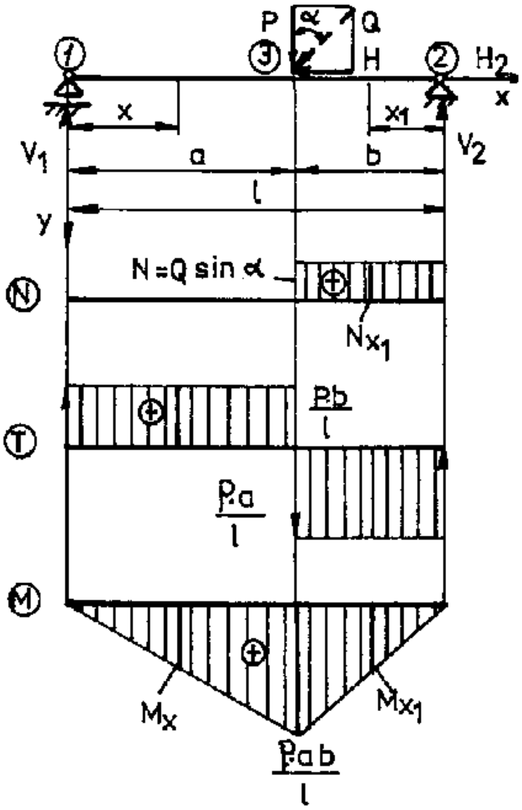


Fig. 2.16

Force Q is deconstructed into the following components:

$P = Q \cdot \cos\alpha$ and $H = Q \cdot \sin\alpha$.

The values of the reactions are:

$H_2 = H = Q \cdot \sin\alpha$; $V_1 = P \cdot b/L$ and $V_2 = P \cdot a/L$.

In an arbitrary section x, located on the left of load Q, the stresses are:

$N_x = 0$; $T_x = V_1 = P \cdot b/L$; $M_x = V_1 \cdot x = P \cdot b \cdot x/L$.

The axial force and the shearing force have constant values,

$N_{1d} = 0$; $T_{1d} = V_1 = P \cdot b/L$,

$M_1 = 0$; $M_{3s} = P \cdot a \cdot b/L$.

Considering the origin in 2 (starting from the right), the following stresses are obtained in section x_1:

$N_{x_1} = H_2 = Q \cdot \sin\alpha$; $T_{x_1} = -P \cdot a/L$,

$M_{x_1} = V_2 \cdot x_1 = P \cdot a \cdot x_1/L$.

The stresses in sections 2 and 3 are:

$N_{2s} = N_{3d} = N_{x_1} = Q \cdot \sin\alpha$;

$T_{2s} = T_{3d} = V_2 = -P \cdot a/L$;

$M_2 = 0$; $M_{3d} = P \cdot a \cdot b/L$.

Notes:

a) The axial force has a constant value, other than zero between the joint and the point of application of the force Q.

b) The shearing force has a constant value, equal to the value of the reaction V_1 on section 1-3, **it rises equally to the value of the vertical component P in the vicinity of force Q**, and on section 3-2 it has a constant value, equal and of opposite direction than reaction V_2.

c) The bending moment proves a linearly variation on both sections (where the shearing forces are constant) and it reaches the maximum value in the vicinity of the point force (where the shearing force passes through zero).

Exercise 2.7. Simply supported beam, subjected to uniformly distributed transverse loads (fig.2.17).

As the loading is symmetrical, the reactions are:

$V_1 = V_2 = p_1 \cdot L/2$.

The stresses in a section *x* are:

$T_x = V_1 - p \cdot x = p \cdot (L/2 - x)$, (linearly variable);

$M_x = V_1 \cdot x - p \cdot x \cdot (x/2) = p \cdot x \cdot (L - x)/2$, (parabolically variable).

The values in the bearings are:

$T_1 = V_1 = p \cdot L/2, \quad M_1 = 0,$

$T_2 = V_2 = - p \cdot L/2, \quad M_2 = 0.$

At distance $x_0 = L/2$; $T = 0$ and therefore

$M_{max} = p \cdot L^2/8$.

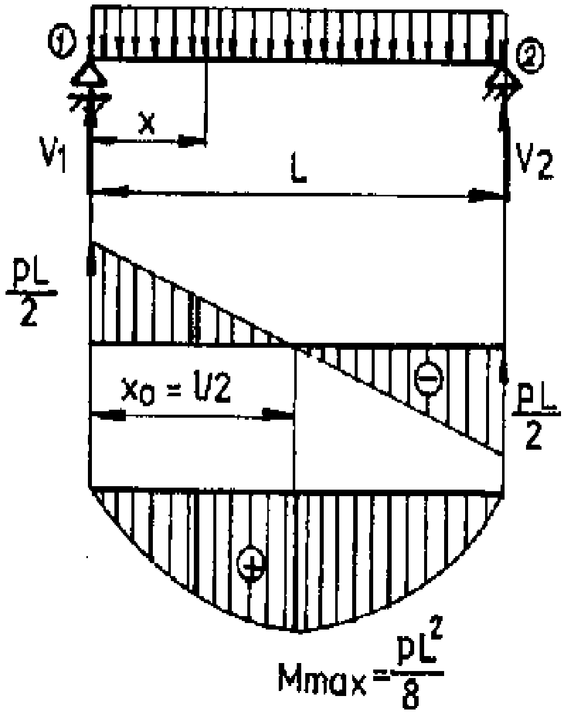


Fig. 2.17

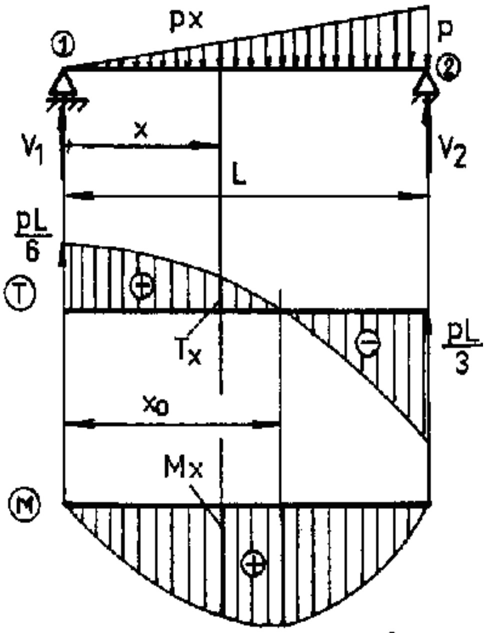
Note:

If we note $P = p \cdot L$, the load on the beam, it is observed that the maximum moment ($M_{max} = p \cdot L^2/8$) is half of the maximum moment produced by the point load P which would act in the middle of the beam, when $M_{max} = P \cdot L/4$ (see fig. 2.17).

Exercise 2.9. Simply supported beam subjected to a linearly variable transverse load (fig.2.18).

The reactions have the following values:

$$V_1 = \frac{1}{L} \cdot \frac{p \cdot L}{2} \cdot \frac{L}{3} = \frac{p \cdot L}{6},$$
$$V_2 = \frac{1}{L} \cdot \frac{p \cdot L}{2} \cdot \frac{2 \cdot L}{3} = \frac{p \cdot L}{3}.$$



$$M_{max} = \frac{p \cdot L^2}{9\sqrt{3}} \quad ;$$

Fig. 2.18

The value of the load in section x is:

$$p_x = p \cdot \frac{x}{L}.$$

The stresses in section x are:

$$T_x = V_1 - \frac{1}{2} \cdot x \cdot p_x = p \cdot \frac{L}{6} - p \cdot \frac{x^2}{2L}, \quad (2^{nd} \text{ degree parabola}),$$

$$M_x = V_1 \cdot x - \frac{1}{2} \cdot x \cdot p_x \cdot \frac{x}{3} =$$
$$= \frac{p \cdot L}{6} \cdot x - \frac{p}{2} \cdot x \cdot \frac{x}{L} \cdot \frac{x}{3} = p \cdot \frac{L^2 - x^2}{6L} \cdot x, \quad (3^{rd} \text{ degree parabola}).$$

The stresses have the following values in the bearings:

$T_{max} = T_1 = V_1 = p \cdot L/6$, $M_1 = 0$, $T_{min} = T_2 = - V_2 = - p \cdot L/3$, $M_2 = 0$.

From this condition:

$$T_x = \frac{p \cdot L}{6} - \frac{p \cdot x_0^2}{2L} = 0,$$

results the coordinate of the section where the bending moment reaches the limit value: $x_0 = \frac{L}{\sqrt{3}} = 0.5574 \cdot L$,

and the maximum moment is: $M_{max} = p \cdot \frac{L^2 - x_0^2}{6} \cdot x_0 = \frac{p \cdot L^2}{9\sqrt{3}}.$

Exercise 2.10. Simply supported beam loaded by torque M_e, (fig.2.19).

The reactions in the bearings are:

$$V_1 = V_2 = \frac{M_e}{L}.$$

The stresses in section x and x_1 are:

$$T_X = T_{X_1} = V_1 = \frac{M_e}{L}, \quad \text{(constant)},$$

$$M_x = -V_1 \cdot x = -M_e \cdot \frac{x}{L}, \text{(linear variation)},$$

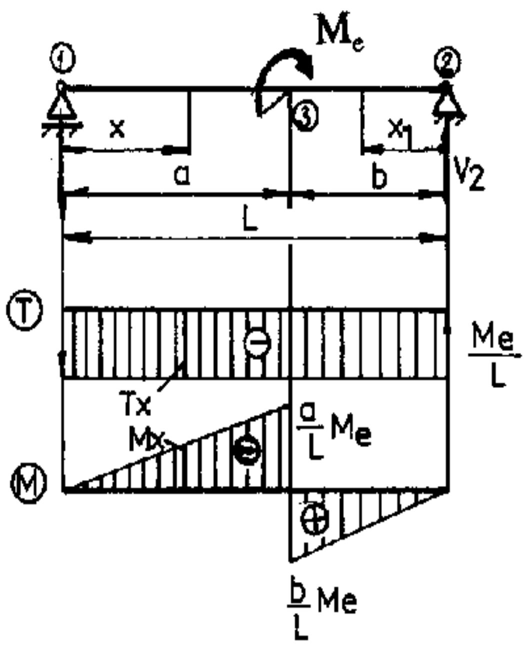


Fig. 2.19

$$M_{x_1} = V_2 \cdot x_1 = M_e \cdot \frac{x_1}{L}$$,(linear variation).

The bending moment is zero in the bearings (x = 0 and x_1 = 0) and reaches the limit values on the left and on the right of section 3 as follows:

$$M_{3s} = -V_1 \cdot a = -\frac{a}{L} \cdot M_e,$$

$$M_{3d} = V_2 \cdot b = \frac{b}{L} \cdot M_e.$$

In the vicinity of the torque, the bending moment diagram rises equally to the value of the torque M_e: $(\mathit{from}\ -\frac{a}{L} \cdot M_e \ \ \mathit{to}\ \ \frac{b}{L} \cdot M_e)$.

2.7.3. Stress diagrams in shafts

The shafts are bars loaded with forces whose directions do not pass through the axis of the bar, or subjected to force couples acting in planes perpendicular to the axis of the bar. The forces or the force couples are transmitted to the shafts through cogwheels, belt wheels, levers, couplings, etc.

The value of the twisting moment is calculated either depending to the distance between the force support and the shaft axis (force arm), or depending on the power and speed to be transmitted.

If a shaft transmits a power P^*, measured in kW, at a speed measured in rotations/minute, then the torque moment results from the relation:

$$\mathbf{P^* = M_t \cdot \omega = M_t \cdot \frac{\pi \cdot n}{30}}, \qquad \text{so that:}$$

$$\mathbf{M_t[kNm] = \frac{30}{\pi} \cdot \frac{P^*[kW]}{n[rot/min]}}. \qquad \mathbf{(2.16)}$$

If power is given in W, the torque moment results in Nm. When power is given in HP (horse power), the torque moment is obtained by using the following relation:

$$M_t[kNm] = 7.02 \cdot \frac{P^*[CP]}{n[rot/\min]}. \quad \textbf{(2.17)}$$

The torque moment is considered to be positive when the twisting moment vector on the left has the direction of the Ox axis, or when it rotates the left section related to the section on the right end in the direction of the right drill.

Exercise 2.11. Draw the power and torque diagrams for a straight shaft (Figure 2.20) that receives a power P* = 10 kW at speed n = 125 rpm through wheel (3) and transmits it like this:

- 25% on wheel (1), - 30% on wheel (2), - and the rest on wheel (4).

The power on the three sections are:

$$P^*_{1-2} = -0.25 \cdot P^* = -2.5\ kW, \qquad P^*_{2-3} = (-0{,}25 + 0{,}3) \cdot P^* = -5{,}5\ \text{kW},$$

$$P^*_{3-4} = (1 - 0{,}25 - 0{,}3) \cdot P^* = 4{,}5\ \text{kW},$$

The variation of the power is shown in diagram P^* in figure 2.24.

The values of the torque on the three sections are:

$$M_{t_{1-2}} = \frac{30}{\pi} \cdot \frac{P_{1-2}}{n} = \frac{30}{\pi} \cdot \frac{-2.5}{125} = -0.191\ kNm,$$

$$M_{t_{2-3}} = \frac{30}{\pi} \cdot \frac{P_{2-3}}{n} = \frac{30}{\pi} \cdot \frac{-5.5}{125} = -0.42\ kNm,$$

$$M_{t_{3-4}} = \frac{30}{\pi} \cdot \frac{P_{3-4}}{n} = \frac{30}{\pi} \cdot \frac{4.5}{125} = -0.344\ kNm.$$

The diagram of variation of the twisting moments M_t, is shown in fig. 2.20.

Note: Taking power through the median wheel and transmitting it to the wheels located on both sides of the drive wheel is one of the most efficient ways of loading the shaft. In this way the power is distributed almost equally both on the left and on

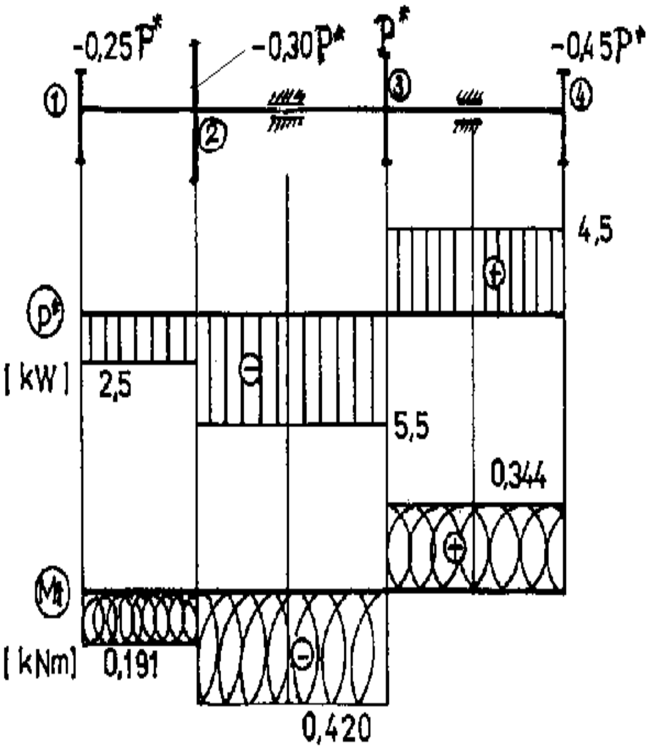


Fig. 2.20

the right of the drive wheel. If the drive wheel is at one of the ends of the shaft, the entire power of 10 kW acts in its vicinity, namely the entire twisting moment, M_t= 0.42 + 0.34 = 0.764 kNm. In this case, the shaft must be adjusted at an almost double twisting moment.

3. GENERAL NOTIONS ON THE THEORY OF ELASTICITY S12

3.1. Introduction

In contrast to the Strength of materials, the **Theory of elasticity aims to determine the state of stress and strain in a body with known elastic characteristics if either the external forces or the shape deformed under the action of these forces are known.**

3.2. Stresses

If an MM is subjected to the action of external forces, additional attraction or rejection forces will appear within it that tend to make it maintain its original shape. If these forces did not exist, the MM would not be able to withstand external loads.

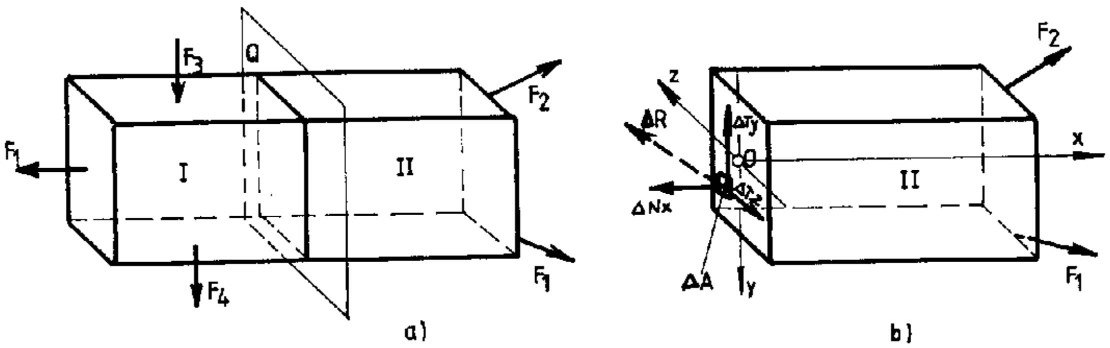


Fig. 3.1

Let us take into consideration a beam, in equilibrium, subjected to a system of external forces (F_1, F_2,..., F_n) (fig. 3.1,a). The external forces tend to change the shape of the beam, whereas the internal forces put up an opposition against the deformation of the beam.

Presumably, the beam had been sectioned by a Q plane, normal on the axis of the bar (Ox). An internal force ΔR will act on each surface element ΔA_x of the separation surface. All the forces ΔR on the entire separation surface keep part I and part II connected to plane Q. The internal force ΔR can be deconstructed into three components parallel to the axes Ox, Oy, and Oz: namely $\mathbf{\Delta N_x}$ $\mathbf{\Delta T_y}$ $\mathbf{\Delta T_z}$.

The strength of the inner force ΔR may differ on the surface and may depend on the position of the area ΔA. The intensity of the force on the area section ΔA is equal to the ratio $\frac{\mathbf{\Delta R}}{\mathbf{\Lambda A}}$. If we reduce the finite area $\mathbf{\Delta A}$ to an infinitesimal area around a point, we obtain a new measure of intensity called **stress**. This is how the normal stress $\boldsymbol{\sigma}_{\mathbf{x}}$ is obtained:

$$\boldsymbol{\sigma}_{\mathbf{x}} = \lim_{\Delta A \to 0} \frac{\Delta N_x}{\Delta A} = \frac{dN_x}{dA}, \qquad \textbf{(3.1,a)}$$

and accordingly, the **tangential stresses:**

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta T_y}{\Delta A} = \frac{dT_y}{dA}, \quad \tau_{yz} = \lim_{\Delta A \to 0} \frac{\Delta T_z}{\Delta A} = \frac{dT_z}{dA}. \qquad \textbf{(3.1,b)}$$

The normal stresses are **positive** if they cause elongation, and **negative** if they cause compression.

The tangential stresses are produced by the forces contained in the Q plane of the section. **They are considered to be positive when they rotate the volume element clockwise and respectively negative when the rotation is counterclockwise**.

The stresses are measured in units of force per unit area Pa, MPa, GPa, N/mm^2, kN/mm^2, etc.

The measures **σ** and **τ are not vectors** (because they are obtained from the ratio of elemental forces to an elemental surface), but they are **tensor measurements** and as such, care must be taken that the operating rules specific to tensors be applied.

Normal stresses are marked with only one index - that of the normal axis to the section, while the tangential stresses are marked with two indices: the first index shows the normal axis to the section, and the second, the axis parallel to the stress.

3.3. Stresses Acting upon a Volume Element

If an infinitesimal element is cut from the beam (fig.3.1) by means of imaginary planes parallel to planes zOy, zOx, xOy, where the distances between them are dx, dy, dz, an elemental parallelepiped is obtained (fig.3.2,a).

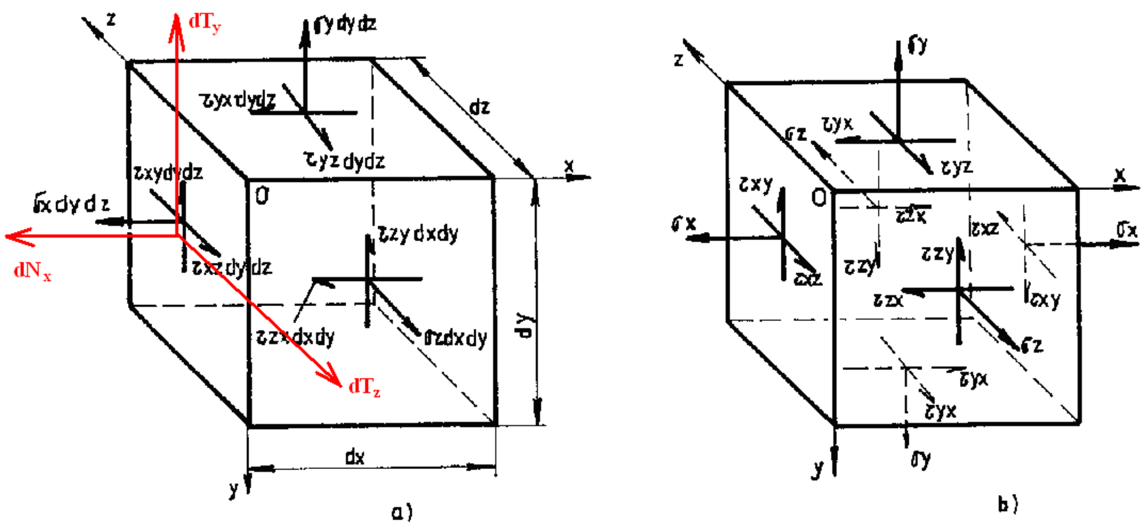


Fig. 3.2

This is considered to be a point from the MM. Stresses **σ_x, τ_{xy}** and **τ_{yz}** determined with the relations (3.1) will act on the left side of this element. The elemental forces on this side are the following:

$$dN_x = \sigma_x \cdot dA = \sigma_x \cdot dy \cdot dz,$$
$$dT_y = \tau_{xy} \cdot dA = \tau_{xy} \cdot dy \cdot dz,$$
$$dT_z = \tau_{xz} \cdot dA = \tau_{xz} \cdot dy \cdot dz.$$

In order to analyze the stresses, we start from the following assumption: **the elemental forces acting on the two elemental areas, of an infinitely small element, that are parallel to each other are also equal and opposite,** that is, if the elemental forces $\boldsymbol{\sigma_x \cdot dA}$, $\boldsymbol{\tau_{xy} \cdot dA}$ and $\boldsymbol{\tau_{xz} \cdot dA}$ load the left side of the element, then the same elemental but opposite forces $\boldsymbol{\sigma_x \cdot dA}$, $\boldsymbol{\tau_{xy} \cdot dA}$ and $\boldsymbol{\tau_{xz} \cdot dA}$ will act on the right side of the element, of the same area dA. **It results that on the sides of the infinitesimal volume element the stresses will act as in figure** (3.2,b).

The 9 components: $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$, $\boldsymbol{\sigma_z}$, $\boldsymbol{\tau_{xy}}$, $\boldsymbol{\tau_{xy}}$, $\boldsymbol{\tau_{yx}}$, $\boldsymbol{\tau_{xz}}$, $\boldsymbol{\tau_{yz}}$, $\boldsymbol{\tau_{zy}}$, fully characterize the state of stress around an arbitrary point O. These are tensor measurements (different from the scalar and vector measurements) and are represented by the **stress tensor**.

$$T_\sigma = \begin{Bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{Bmatrix}. \qquad \textbf{(3.2)}$$

The stress tensor is a second order tensor, which contains the above-mentioned 9 components on the 6 sides of the volume element. On each side of the volume element there are one component $\boldsymbol{\sigma}$ parallel to the normal axis to the side and two components $\boldsymbol{\tau}$, contained in the plane of the section and parallel to the two axes of the section.

The infinitesimal element subjected to the elemental forces is in equilibrium, and therefore the normal forces must be two by two collinear, equal in size and opposite, and the system of tangential forces must also be in equilibrium. Thus, the tangential forces (T_y) must be equal in size and opposite in direction, paired and the moment at the center of the element must be zero:

$$\mathbf{2 \cdot \boldsymbol{\tau}_{xy} \cdot dy \cdot dz \cdot \frac{dx}{2} - 2 \cdot \boldsymbol{\tau}_{yx} \cdot dx \cdot dz \cdot \frac{dy}{2} = 0.}$$

By simplifying with **dx·dy·dz**, the following will result:

$$\boldsymbol{\tau}_{\mathbf{xy}} = \boldsymbol{\tau}_{\mathbf{yx}} \ .$$

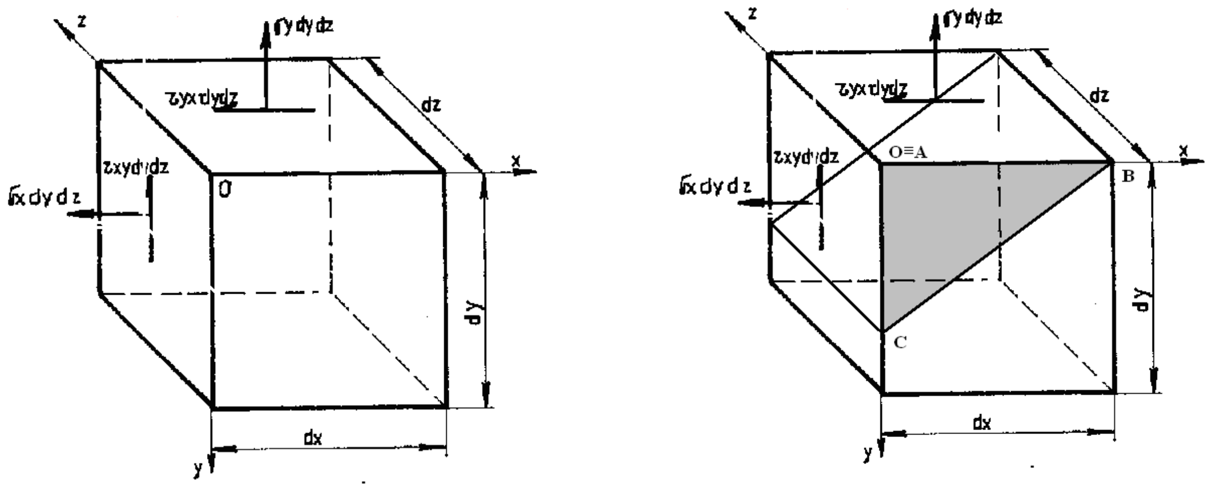
If we set similar conditions for the stresses on the other sides, parallel to each other, from figure (3.2, b), we obtain the relations:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \text{ and } \tau_{zx} = \tau_{xz}. \qquad \textbf{(3.3)}$$

These relations represent **the duality of the tangential stresses** and specify that: **on the perpendicular sides of an infinitesimal element, the following tangential stresses τ_{xy} and τ_{yx} can exist simultaneously. They are contained in planes that correspond to the sides of the volume element and, in pairs, they cause couples of equal size and opposite direction. Therefore, they must be symmetrical to the common edge of the two sides.** Relations (3.3) prove that out of the 9 components of tensor (3.2) only 6 are distinct and therefore **the stress tensor is symmetrical to the main diagonal**.

3.4. Plane State of Stress. Variation of Stresses around a Point.

In many engineering problems, the particular case of the general state of stress is encountered, when the MM is loaded with coplanar forces in equilibrium, in this case on the unloaded surface, and there are no normal and parallel loads. Also, taking into account the equilibrium condition, the forces will be zero on a side which is parallel to the first and located at infinitely small distance (dz). In this case all the forces are coplanar and the corresponding state of stress is called a **plane state of stress** and it can be represented as in figure (3.3a,b).



a) b)

Fig. 3.3

We take into consideration the infinitely small element in figure 3.4 under the form of a triangular prism, based on a rectangular triangle, cut from the volume element in figure (3.3,b) and subjected to components σ_x, σ_y , $\tau_{xy} = \tau_{yx}$. On the AC side, inclined with angle α, the unknown stresses σ_α and τ_α will occur.

The BC side is considered to be of area **dA**, and the thickness of the element is constant. In this case, the area of the AC side is **dA·cosα,** and that of AB side is **dA·sinα**.

From the equations of the projections of the elemental forces on directions σ_α and τ_α, from the equilibrium conditions of the element, the following are obtained:

$$\begin{aligned}&\sigma_\alpha \cdot dA - \sigma_x \cdot dA \cdot \cos^2\alpha - \sigma_y \cdot dA \cdot \sin^2\alpha - \\ &\quad -\tau_{xy} \cdot dA \cdot \cos\alpha \cdot \sin\alpha - \tau_{xy} \cdot dA \cdot \sin\alpha \cdot \cos\alpha = 0\end{aligned}$$

$$\begin{aligned}&\tau_\alpha \cdot dA + \sigma_x \cdot dA \cdot \cos\alpha \cdot \sin\alpha - \sigma_y \cdot dA \cdot \sin\alpha \cdot \cos\alpha - \\ &\quad -\tau_{xy} \cdot dA \cdot \cos^2\alpha + \tau_{xy} \cdot dA \cdot \sin^2\alpha = 0\end{aligned}$$

By simplifying with **dA** and considering that $\tau_{xy} = \tau_{yx}$, the following results:

$$\sigma_\alpha = \sigma_x \cdot \cos^2\alpha + \sigma_y \cdot \sin^2\alpha + \tau_{xy} \cdot \sin\alpha \cdot \cos\alpha$$
,

$$\tau_\alpha = \left(-\sigma_x + \sigma_y\right) \cdot \sin\alpha \cdot \cos\alpha + \tau_{xy} \cdot \left(\cos^2\alpha - \mathrm{si}\right.$$
.

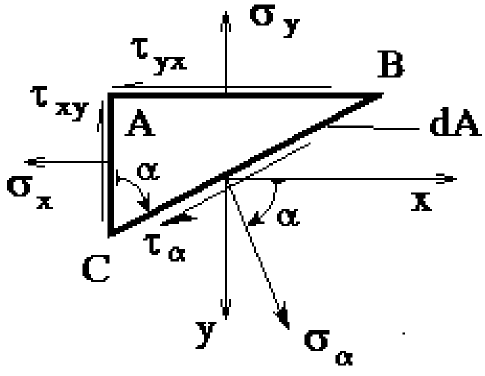


Fig. 3.4

Taking into account that:

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}, \quad \cos^2\alpha = \frac{1 + \cos 2\alpha}{2}, \quad \sin\alpha \cdot \cos\alpha = \frac{\sin 2\alpha}{2},$$

the above expressions become:

$$\sigma_\alpha = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy} \cdot \sin 2\alpha, \qquad \textbf{(3.4,a)}$$

$$\tau_{\alpha} = -\frac{\sigma_x - \sigma_y}{2} \cdot \sin 2\alpha + \tau_{xy} \cdot \cos 2\alpha \,. \qquad \textbf{(3.4,b)}$$

These relations allow the determination of stresses on a surface inclined at angle α. The normal to this surface makes angle α with axis Ox. The angle α can also be defined as the angle at which the Ox axis must be rotated in order to overlap the normal to the given inclined surface.

Angle α is considered to be positive when it rotates the Ox axis clockwise to the normal to the inclined surface and negative when it rotates it counterclockwise.

Relations (3.4) show that stresses σ_{α} and τ_{α} are circular functions of parameter 2α. Since it is necessary to know the maximum and minimum values of the stresses, the expressions (3.4,a) and (3.4,b) are derived in relation to parameter 2α. The limit values of the stresses are obtained for the value of parameter α for which the derivative is canceled.

$$\frac{d\sigma_{\alpha}}{d(2\alpha)} = -\frac{\sigma_x - \sigma_y}{2} \cdot \sin\alpha_{1,2} + \tau_{xy} \cdot \cos\alpha_{1,2} = 0$$

It is noted that the derivative of σ_{α} is τ_{α} and therefore **on the sides where σ_{α} reaches limit values, the tangential stresses are null**.

The planes on which the tangential stresses are null are called **main planes** and the normals to these planes are called **main directions**.

The normal stresses on the main planes are called **main stresses** and therefore the main stresses are maximum or minimum stresses, on the planes where $\tau = 0$, that is for:

$$\mathrm{tg}2\alpha_{1,2} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \text{ or } \alpha_{1,2} = \frac{1}{2} \cdot \mathrm{arctg}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \frac{\pi}{2}. \qquad \textbf{(3.5)}$$

The above relations have two indices because the tangent function has the period π and thus there will be two solutions on a whole circle, namely $2\alpha_1$ and $2\alpha_2$, different by 180^{o} and therefore the directions α_1 differ from α_2 by 90^{o}; that is, they are perpendicular to each other.

In order to obtain angle $\boldsymbol{\alpha_1}$ the following relation can also be used:

$$\alpha_1 = \text{arctg}\frac{\tau_{xy}}{\sigma_x - \sigma_2}. \qquad \textbf{(3.5,a)}$$

Direction α_1 is for the maximum stress σ_1 and direction α_2 is for the minimum stress σ_2.

If we take the trigonometric relations into consideration in expressions (3.5), we get:

$$\sin 2\alpha_{1,2} = \frac{\text{tg}2\alpha_{1,2}}{\pm\sqrt{1+\text{tg}^2 2\alpha_{1,2}}} = \frac{\tau_{xy}}{\pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}},$$

$$\cos 2\alpha_{1,2} = \frac{1}{\pm\sqrt{1+\text{tg}^2 2\alpha_{1,2}}} = \frac{\frac{\sigma_x - \sigma_y}{2}}{\pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}.$$

By replacing these expressions in expression (3.4,a), we get the expressions of the two main stresses:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \qquad \textbf{(3.6)}$$

The maximum stress $\sigma_1$is obtained when the plus sign appears in front of the radical, while the minimum stress σ_2 is obtained when there is the minus sign.

Doing the same with the second relation (3.4,b), by derivation with respect to parameter 2α and equalizing to zero, we obtain the values for which stress $\boldsymbol{\tau_\alpha}$ reaches the limit:

$$\frac{d\tau_\alpha}{d(2\alpha)} = -\frac{\sigma_x - \sigma_y}{2} \cdot \cos 2\alpha'_{1,2} - \tau_{xy} \cdot \sin 2\alpha'_{1,2} = 0,$$
$$\text{tg}\alpha'_{1,2} = \frac{\sigma_y - \sigma_x}{2 \cdot \tau_{xy}} = -\frac{1}{\text{tg}\alpha_{1,2}}. \qquad \textbf{(3.7)}$$

From relation (3.7) results that directions $2\alpha_{1,2}$ and $2\alpha'_{1,2}$ are perpendicular, so it follows that: **the limit tangential stresses are on those sides of the element which differ by 45° from the sides on which the main normal stresses occur.**

If we replace the parameter $2\tilde{\alpha}_{1,2}$ in expressions (3.4), it results:

$$\boldsymbol{\sigma_m = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_\alpha + \sigma_{\alpha+90^\circ}}{2} = ct.}, \tag{3.8}$$

$$\boldsymbol{\tau_{1,2} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}. \tag{3.9}$$

Relation (3.8) shows that the sum of the normal stresses on two perpendicular sides is constant.

Relation (3.9) expresses the equality between the semi-difference of the main normal stresses with the maximum tangential stress and with the value below the radical in relation (3.6) and can be written:

$$\boldsymbol{\sigma_{1,2} = \sigma_m \pm \tau_1}. \tag{3.10}$$

On the sides inclined at 45° to the main planes, extreme tangential stresses and normal average stresses occur, equal to the half-sum of the normal stresses.

3.5. Strains and Displacements

The state of stress was analyzed as an effect of the internal forces and, similarly, the change of the dimensions will be further analyzed.

A **strain is a modification of the size of the MM**. The modification of the length is called **elongation**, when the MM is stretched and respectively, it is called **shortening**, when the MM is compressed. The elongations and the shortenings are marked with Δl, Δx, Δy, Δz, etc.

An **angular deformation** defines the modification of the (right) angles and it is marked with $\Delta\varphi$; $\Delta\theta$, etc.

In order to simplify and make the study of strains more clear, we shall consider a plane element OABC cut from an in-plane loaded MM. The plane state of stress can be understood as the overlapping of three states of stress: two states of normal stress (fig.3.5, b and c) and one of pure shear (fig.3.5, d). Each of these states of stress produces strains.

The state of stress in figure (3.5, b) changes the length of the member, so that the member with the initial dimensions (dashed line) changes and takes the form of

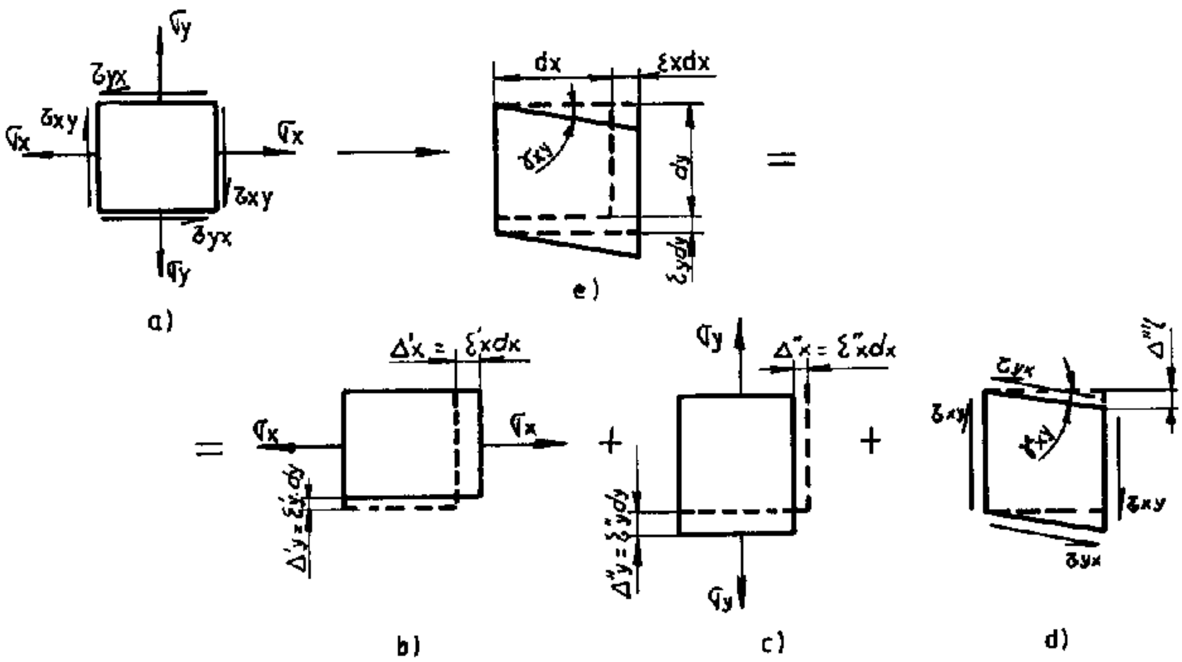


Fig. 3.5

the member represented by a thick line. These changes are **linear strains**, Δ~x and Δ ~y - where Δ~x is an elongation, and Δ~y a contraction. Linear strains are measured in mm or μm.

The member deforms similarly under the stress in figure (3.5,c), with the elongation Δ~~y and the contraction Δ~~x.

Because linear strains cannot characterize properly the strains of an MM, because they depend on its dimensions, the notions of **specific strains** are used.

A specific linear strain in a certain direction is defined as the ratio between the elongation (shortening) of the member and its initial length in the respective direction. The following **specific elongations** are obtained for the members in figure (3.5, b, c):

$$\varepsilon_x^{'} = \frac{\Delta' x}{dx} \quad \text{and} \quad \varepsilon_y^{''} = \frac{\Delta'' y}{dy}, \qquad \textbf{(3.11,a)}$$

and the following **specific shortenings (contractions)**:

$$\varepsilon_y^{'} = \frac{\Delta' y}{dy} \text{ and } \varepsilon_x^{''} = \frac{\Delta'' x}{dx} \qquad \textbf{(3.11,b)}$$

The tangential stresses deform the member as in figure (3.5,c). Under the action of the tangential stresses, the member changes only its right angle, and the lengths of the sides remain the same. The modification of the right angle is marked with γ_{xy}.

Since the angle γ_{xy} is very small, the **specific angular deformation** can be defined as follows:

$$\gamma_{xy} \approx tg\, \gamma_{xy} = \frac{\Delta''' l}{dx}, \qquad \textbf{(3.12)}$$

and it is called **specific sliding**.

The specific linear and angular deformations are dimensionless. In the technical literature, the specific elongations are given in μm/m or in percentage %, and the specific sliding can be expressed in μm/m or in radians.

The specific strains are tensors just like the stresses.

The path covered by a point of the MM from its initial position, corresponding to an unloaded MM, to the final position, after loading, is called a displacement. The displacements are vector quantities.

The displacement can usually be the result of the following four general situations:

a) the shifting of the whole MM;

b) the rotation of the whole MM;

c) the change of the dimensions of the MM;

d) the modification of the MM’s angles.

The first two are displacements of the rigid body, and the last two types are caused by the deformation of the MM. The displacements of the rigid have been studied in kinematics. In Strength of materials, only the displacements caused by the deformation of the MM will be studied.

4. THE MECHANICAL BEHAVIOR OF THE MECHANICAL MEMBERS – S13

4.1. The Physical Aspect

The analysis of stresses and that of strains were studied separately, independently of each other and without taking into account the physical-mechanical characteristics of the material the MM is made of. In reality, however, the stresses and strains depend on each other and the interdependency is directly dependent on the physical and mechanical properties of the material of the MM (Mechanical Member).

In the *Strength of Materials*, the state of stress and that of strain of the bodies in equilibrium are analyzed. The equilibrium in Strength of materials, called **static equilibrium**, differs from the equilibrium in mechanics which implies zero acceleration. An MM subjected to forces, in equilibrium, is deformed and thus **some of its parts will displace in relation to others**. The movement will be accelerated until a certain strain is reached. **The straining process will end when the internal forces, caused by strain, become large enough to balance the action of the external forces**. When this stage is reached, the MM will be in equilibrium again. **If the internal forces will not be so powerful as to stop the strains, the MM will fracture.**

The load is called static if the forces are applied in such a way that the increase of the strains is small and it can be assumed that the acceleration effect is neglectable during the deformation process. Such a process is called **a quasi static process.** We will hereafter refer to the quasi static process produced by loads as a **static loading**.

The physical aspect in the *Strength of Materials* represents the relations between stresses and strains. These relations, as well as the physical and mechanical properties of the materials, are established experimentally (through mechanical tests).

4.2. The Tensile Test – S14

4.2.1. The specimen

The relation between stresses and strains can be more easily and conveniently established on a long MM in which there is a uniaxial state of stress. For this purpose, we consider a specimen (fig.4.1) which is axially loaded, at the two ends, by forces F (fig. 4.1, a). The uniaxial state of stress is observed on the volume element sectioned from the beam (fig. 4.1, c).

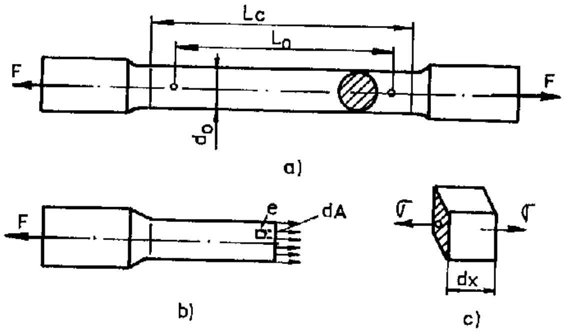


Fig. 4.1

The equilibrium equation for the left side of the specimen (fig. 4.1, b) is the following:

$$\mathbf{F} - \int_{\mathbf{A}} \boldsymbol{\sigma} \cdot \mathbf{dA} = 0.$$

Accepting **the assumption that the normal stresses are uniformly distributed over the entire section** (σ = ct.) the above equilibrium equation results in $F = \sigma \cdot A_0$ which leads to the following:

$$\boldsymbol{\sigma} = \frac{\mathbf{F}}{\mathbf{A_0}}. \qquad \textbf{(4.1)}$$

The tensile test can be performed on a cylindrical steel specimen as shown in figure (4.1, a), according to SR EN 10002-1; 1994. It has the same diameter on the **calibrated length L_c**. Two marks are made on this length at distance L_0, called the **length between marks**. **The length of the specimen** is considered to be the length between marks L_0.

The elongation of member dx is:

$$\mathbf{\Delta dx = \varepsilon \cdot dx,}$$

and the elongation of the specimen (between the two marks) will be:

$$\mathbf{L = \int_0^{L_0} \Delta dx = \int_0^{L_0} \varepsilon \cdot dx.}$$

Starting from the assumption that the specific length is the same throughout the calibrated length (ε = ct.), the above relation leads to:

$$\mathbf{\Delta L = \varepsilon \cdot L_0; \ \varepsilon = \frac{\Delta L}{L_0}}. \qquad \mathbf{(4.2)}$$

4.2.2. The testing machine and the measuring devices

The ends of the specimens have various shapes, chosen according to the fasteners of **the testing machine**. The testing machine is a special presser which ensures the slow increase of the axial force F and the precise measurement of its value under the prescribed loading speed conditions.

The elongation of the specimen (between the marks) is measured, with a gauge called **extensometer**, concurrently measuring the axial force. The extensometer is fastened on the specimen by means of two pairs of knife edges: one pair is fixed and the other one is movable. They are attached to the specimen in the vicinity of the marks (at distance L_0).

4.2.3. The tensile test diagram

During the increase of the load, the intermittent values of the load and of the elongation are simultaneously read. Many laboratories are equipped with installations that record the force-elongation diagram. **The tensile test diagram** F = f(Δl), recorded by the gauge or represented according to the measurements, for light steel, has the form shown in figure (4.2, a). In order to obtain the diagram σ = f(ε),relations (4.1) and (4.2) are used; load F is divided at **the initial area A_0** and the elongation ΔL at **the initial length L_0**. Graphically representing the obtained data, in the system of axis: the abscissa - **the specific elongations** ε and the ordinate - **the stresses σ,** the **characteristic curve of the material** is obtained. For steel, this looks like the one in figure (4.2,b).

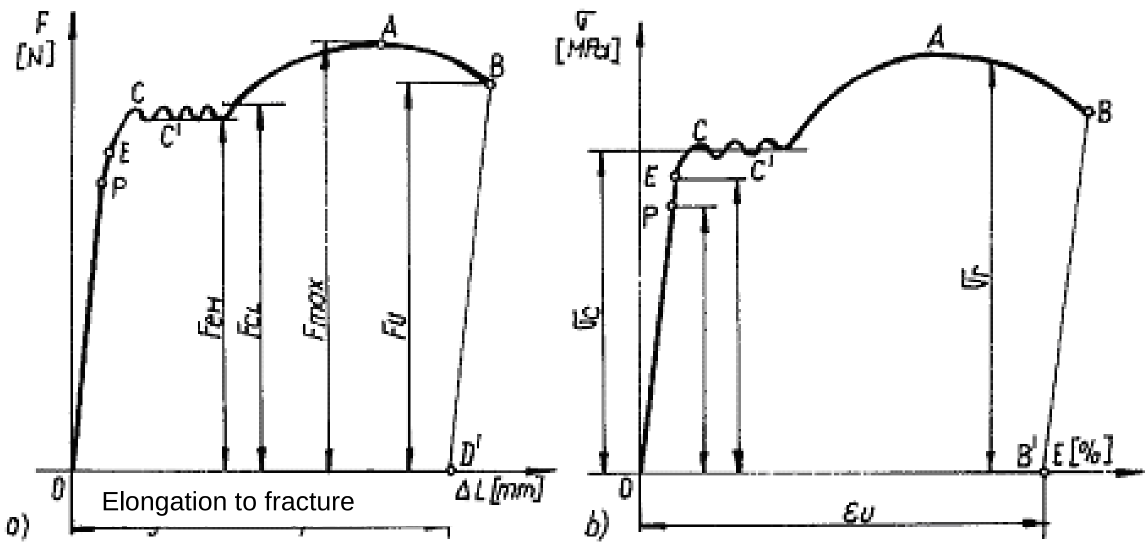


Fig. 4.2

Only a part of the characteristic curve, namely, the OPECC′A is used for calculating strength.

4.3. The Elastic and Mechanical Properties of the Materials

The characteristic curve has a series of special points, called limits, which define the following characteristic measurements:

a) The limit of proportionality, marked on the curve by point P, is the maximum stress up to which there is **linearity between stresses and deformations** ($\sigma_p = \frac{F_p}{A_0}$).

The equation of the area of proportionality (of section OP) is:

$$\boldsymbol{\sigma = E \cdot \varepsilon,} \qquad \textbf{(4.3)}$$

and it is called **Hooke's Law**. This shows that, **up to the proportionality limit, the specific elongations are proportional to the stresses**.

Property E is called the **longitudinal modulus of elasticity** (Young's modulus). Each material has a unique value of this property, which is a **measure of the rigidity** of that material. Thus, steels, regardless of their quality, have on average: $E_{OL} \cong 210$ GPa, and aluminum $E_{AL} \cong 75$ GPa.

The values of the elasticity modules and the elastic characteristics for different materials are given in the tables (see annex 2).

Only two materials have the characteristic curve with an area of proportionality, and those are **steel** and **wood**. **They obey Hooke's law**. The other materials have curvilinear characteristics. Because it is useful to use Hooke's law in these materials, SR EN 10002-1,2; 1994 defines the specific terms for the modulus of elasticity.

b) The conventional linear modulus of elasticity, which is the ratio between the stress and the corresponding specific elongation, for metals having a linear elastic section of the tensile characteristic curve, is the following:

$$\mathbf{E = \frac{\sigma}{\varepsilon}}. \qquad \mathbf{(4.4)}$$

For other materials, it is necessary to consult SR EN 10002-1,2; 1994.

c) **The elastic limit**, marked on the characteristic curve by point E (fig.4.2, b), is the value of the maximum stress up to which the material is perfectly elastic;

$$\boldsymbol{\sigma_e = \frac{F_E}{A_0}}. \qquad \mathbf{(4.5)}$$

Experience has shown that there is no perfectly elastic material, that is, after unloading, it does not return to the original length. All the materials, **even subjected to a relatively small load, show a permanent deformation**. The value of this deformation depends on the strength of the applied load.

d) The (apparent) flow limit, marked on the characteristic curve by point C (fig.4.2, b) is the value of the stress at which the elongation increases although the load is maintained almost constant (fig.4.2,b):

$$\boldsymbol{\sigma_c = \frac{F_c}{A_0}}. \qquad \textbf{(4.6)}$$

In SR EN 10002-1; 1994 the flow limit is noted also with R_c

After reaching the flow limit, the specimen continues to deform plastically, without a stress increase. The characteristic curve has an oscillating path, between **the upper flow limit σ_{cs}** and **the lower flow limit σ_{ci}.** The average value of the oscillations can be approximated by a straight line, which is called the **flow level CC′** (fig.4.2). The plastic deformation that occurs for the flow level (CC′), in light steel, is 20 to 50 times higher than in the elastic one (the abscissa of point E).

The plastic deformation during the flow occurs as a result of the relative siding between the faults formed and inclined at 45° to the axis of the specimen, without weakening the cohesion between the faults.

For this reason, upon reaching the flow limit, fine inclined lines appear, of a darker color, at 45° to the axis of the specimen, called **Lüders - Chernov lines**. These lines multiply into strips, which widen progressively until they cover the entire calibrated section of the specimen. The lines represent the traces of the siding planes of the material, where the tangential stresses are maximum ($\tau_{max} = \sigma_c / 2$).

Once the Lüders lines cover the entire calibrated section of the specimen, the stress begins to increase concurrently with the deformation. On the characteristic curve, this section is represented by the CA curve (fig.4.2) and is called the **hardening area**.

If from a point on this area, instead of continuing the loading, it slowly unloads from point M, a linear relation between σ and ε is obtained during unloading. The section MO′ is a line parallel to OP (fig.4.2, b). When reloading the specimen, the line O′M is covered, so that the material behaves elastically up to point M. Thus, point M is the new elastic limit of the material, higher than the one determined at the beginning. This operation of increasing the limits $\sigma_p = \sigma_E = \sigma_c = \sigma_M$ is called **strain hardening.**

e) The fracture strength of the material, marked on the characteristic curve by point A (fig.4.2, b) is the maximum value of the stress and is marked with σ_r (R_m in SR EN 10002-1; 1994)

$$\sigma_r = \sigma_{max} = \frac{F_{max}}{A_0},$$

where;

$$A_0 = \frac{\pi \cdot d_0^2}{4}$$ is the initial section area.

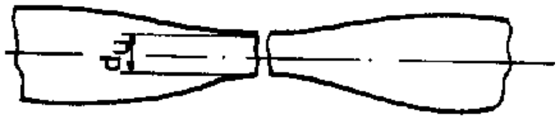


Fig. 4.3

f) In specimens made of light (ductile) steel, when the load approaches F_{max}, the **constriction of the specimen** occurs (fig.4.3). In the constriction area, the section decreases until sudden noisily fracture occurs. After the constriction, load F applied to the specimen decreases in intensity, which is represented on the characteristic curve through area AB (fig.4.2).

By measuring the diameter of the specimen subjected to an arbitrary load on the section AB (after the occurrence of the constriction) and calculating the corresponding area, **the specific constriction** can be determined**.**

$$\psi = \frac{A_0 - A}{A_0}. \qquad \textbf{(4.8,a)}$$

For a fractured specimen, the rupture constriction is:

$$Z = \frac{A_0 - A_u}{A_0} \cdot 100[\%] \qquad \textbf{(4.8,b)}$$

where:

$$A_u = \frac{\pi \cdot d_u^2}{4}$$ is the area of the fracture section.

g) By placing the two pieces of the fractured specimen end to end, we can measure the **final length between the marks**, L_u and **the specific fracture elongation** can be determined (according to SR EN 10002-1; 1994)**;**

$$\mathbf{A_r = \varepsilon_r = \frac{L_u - L_0}{L_0} = \frac{\Delta L_u}{L_0}}. \qquad \textbf{(4.9)}$$

h) Experimentally, it has been shown that with the elongation of a beam (specimen) a decrease of the section occurs, called **cross-sectional contraction**. It has been found that for the linear-elastic domain this contraction is proportional to the specific elongation. As such, a specific elongation of the specimen with ε_x corresponds to a cross-sectional contraction proportional to the elongation ε_x;

$$\boldsymbol{\varepsilon_{tr} = \varepsilon_y = \varepsilon_z = -\nu \cdot \varepsilon_x},$$

where:

ν - is the **cross-sectional contraction coefficient** or **Poisson's coefficient**. Poisson's coefficient is an elastic constant of material. Its value ranges between 0.16 and 0.42 and is given in the tables. If the deformation is plastic, the body does not change its volume and $\nu = 0.5$.

The measurements: **the flow limit (σ_c), the fracture strength (σ_r), the fracture elongation (ε_r), and the rupture constriction (Z) are called mechanical properties of the material.** The constants: **the longitudinal modulus of elasticity (E), the cross-sectional contraction coefficient (ν), the limit of proportionality (σ_p), the elastic limit (σ_e) are called elastic properties of the material.**

For OL 37 the mechanical and elastic properties, according to STAS 1500-75, are the following:

$\sigma_r = 370...450\text{MPa}$	$E = 210\text{GPa}$
$\sigma_c = 210...240\text{MPa}$	$\nu = 0{,}24...0{,}28$
$\varepsilon_r = 25...26\%$	$\sigma_e \cong \sigma_p = 200\text{MPa}$
$Z = 60...70\%$	

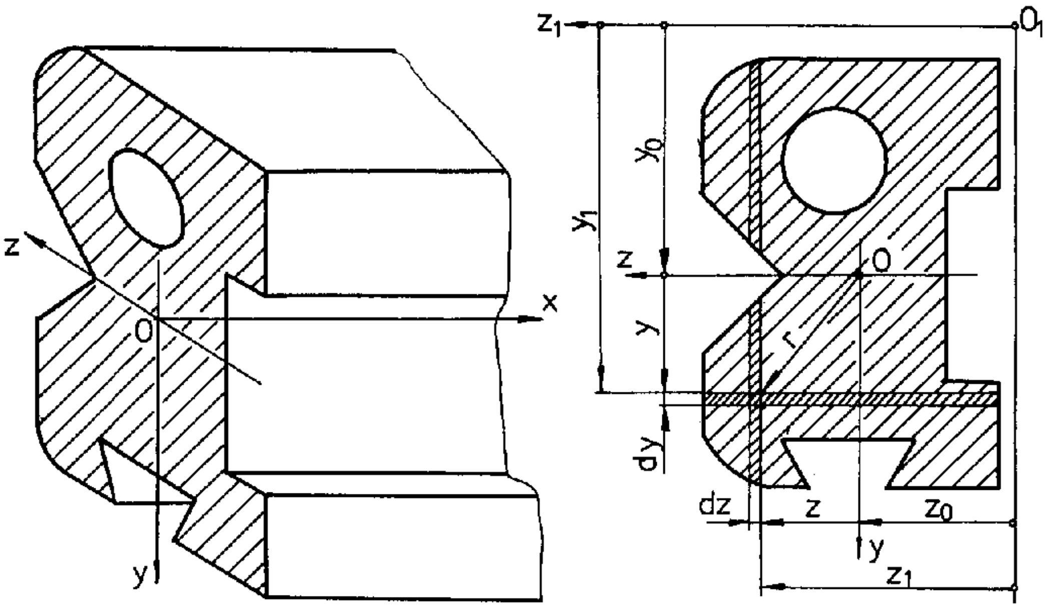
5. GEOMETRICAL PROPERTIES OF THE CROSS SECTIONS – S15

5.1. General Notions

In the calculations made in Strength of Materials, we use **quantities that depend on the shape and dimensions of the beam's cross section**. These are called **geometrical properties of the section** and are the following: the area, the static moments, the moments of inertia, the strength modulus and the radius of gyration.

In order to study these quantities, **the bear is imaginary sectioned with a normal plane (perpendicular) on the longitudinal axis** (a cross section) and **a system of straight three-orthogonal axes is used, with axis Ox along the beam, having the origin in the center of gravity of the section and with axes Oy and Oz in its plane** (fig.5.1). Because the origin of the system is in the center of gravity of the section, **axes Oy and Oz are called central axes**.

Appendix 4 contains the formulas for calculating the geometrical properties of certain cross sections frequently used in Strength of Materials.



a) b)

Fig. 5.1

5.2. Area of the Cross Section

An area element $dA = dy \cdot dz$ can be considered around a point in the section plane. However, other formulas will also be used for the area element from here on: dA = b·dy, respectively dA=h·dz for rectangles, or dA = 2π·r·dr for circles, etc. The area of the cross section will be obtained with the relation:

$$\mathbf{A} = \int_{A} \mathbf{dA}\,. \qquad \mathbf{(5.1)}$$

The areas of the cross sections of the standardized beams (profiles) are listed in the tables in the annexes. Formula (5.1) will be used to determine the areas of arbitrary cross sections.

5.3. Static Moments

Strength of Materials uses static moments of the surfaces with respect to axes z and y, defined by the expressions:

$$\mathbf{S_Z} = \int_{A_1} \mathbf{y \cdot dA}, \quad \mathbf{S_y} = \int_{A_2} \mathbf{z \cdot dA}\,, \qquad \mathbf{(5.2)}$$

where A_1 and A_2 are parts of area A. The static moments of the whole section with respect to the axes y_1 and z_1, parallel to the central axes y and z, are:

$$\mathbf{S_{z_1}} = \int_{A} \mathbf{y_1 \cdot dA}, \quad \mathbf{S_{y_1}} = \int_{A} \mathbf{z_1 \cdot dA}\,,$$

where $\mathbf{y_1 = y_0 + y,\ z_1 = z_0 + z}$ (fig. 5.1,b).

By applying the theorem of the static moment (of Varignon),

$$\int_{A} \mathbf{y_1 \cdot dA} = \mathbf{y_0} \cdot \int_{A} \mathbf{dA}, \quad \int_{A} \mathbf{z_1 \cdot dA} = \mathbf{z_0} \cdot \int_{A} \mathbf{dA}, \qquad \mathbf{(5.3,a)}$$

we obtain the formulas that define the position of the center of gravity with respect to the initially chosen system of axis $O_1y_1z_1$:

$$\mathbf{y_0 = \frac{\int_A y_1 \cdot dA}{\int_A dA} = \frac{\sum y_i \cdot A_i}{\sum A_i}, \qquad z_0 = \frac{\int_A z_1 \cdot dA}{\int_A dA} = \frac{\sum z_i \cdot A_i}{\sum A_i}} \qquad \mathbf{(5.3)}$$

The static moments of the whole section are null with respect to the central axes:

$$\mathbf{S_Z = \int_A y \cdot dA = 0, \quad S_y = \int_A z \cdot dA = 0}. \qquad \mathbf{(5.4)}$$

Due to the fact that the axes of symmetry are also central axes, the static moments of the whole section with respect to these axes are null. Obviously, **the static moment for a part of the section, relative to the axes of symmetry, is not null.**

Static moments are measured in mm^3, cm^3, m^3.

5.4. Moments of Inertia

5.4.1. Defining relations

The following geometrical moments of inertia are defined:

a) axial to the axis Oz, and respectively Oy (fig. 5.1,b):

$$I_Z = \int_A y^2 \cdot dA, \quad I_Y = \int_A z^2 \cdot dA,, \qquad \mathbf{(5.5)}$$

b) centrifugal (in plane Ozy):

$$I_{zy} = \int_A y \cdot z \cdot dA, \qquad \mathbf{(5.6)}$$

c) polar (to the center of gravity O):

$$\mathbf{I_o = I_P = \int_A r^2 \cdot dA. = I_z + I_y}. \qquad \mathbf{(5.7)}$$

Since $r^2 = y^2 + z^2$, relation (5.7) leads to:

$$\mathbf{I_P = \int_A \left(y^2 + z^2\right) \cdot dA = \int_A y^2 \cdot dA + \int_A z^2 \cdot dA = I_z + I_y}$$

Thus, **the polar moment of inertia is equal to the sum of the axial moments of inertia, relative to the orthogonal axes passing through the considered pole**.

Because the area element is a positive size, and z^2, y^2 are r^2 are positive sizes, it follows that **the axial and polar moments of inertia are strictly positive.**

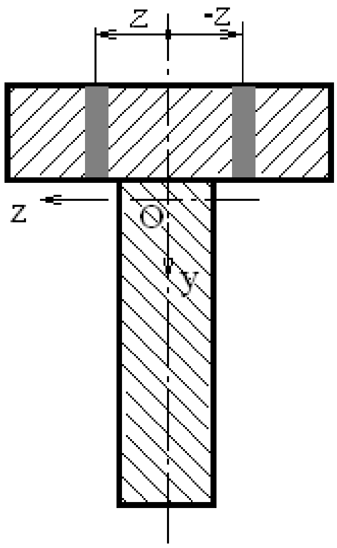


Fig. 5.2

The centrifugal moment of inertia is the sum of the area element dA and two coordinates (y, z) and as such can be positive, negative or equal to zero. For sections with at least one axis of symmetry (Oy axis in figure 5.2), there are always, on the y-ordinate, two area elements symmetrically relative to the axis of symmetry (Oy): one of positive sign (+ z) and the other negative. (-z) so that, for the entire area of the section, the following is obtained:

$$\mathbf{I_{zy} = \int_A z \cdot y \cdot dA = 0}.$$

Thus, **the moment of inertia which is centrifugal to a system of axes of which at least one is the axis of symmetry is null**.

The moments of inertia are measured in mm^4, cm^4, m^4.

5.4.2. Variation of moments of inertia with respect to parallel axes

For the section in figure (5.1, b), the axial moments of inertia $\mathbf{I_z}$, $\mathbf{I_y}$ and the centrifugal $\mathbf{I_{zy}}$ relative to the central system of axes Ozy are known.

The area element dA, in the system of axes $O_1z_1y_1$, parallel to Ozy (fig.5.1, b), has the following coordinates:

$$\mathbf{y_1 = y_0 + y}, \qquad \mathbf{z_1 = z_0 + z}.$$

In relation to the system of axes $O_1\ y_1\ z_1$, the moments of inertia have the following expressions:

$$I_{z_1} = \int_A y_1^2 \cdot dA = \int_A (y + y_0)^2 \cdot dA = \int_A y^2 \cdot dA + y_0^2 \int_A dA + 2 \cdot y_0 \int_A y \cdot dA,$$

$$I_{y_1} = \int_A z_1^2 \cdot dA = \int_A (z+z_0)^2 \cdot dA = \int_A z^2 \cdot dA + z_0^2 \int_A dA + 2 \cdot z_0 \cdot \int_A z \cdot dA,$$

$$I_{z_1 y_1} = \int_A y_1 \cdot z_1 \cdot dA = \int_A (y+y_0) \cdot (z+z_0) dA =$$
$$= \int_A y \cdot z \cdot dA + y_0 \cdot z_0 \cdot \int_A dA + y_0 \cdot \int_A z \cdot dA + z_0 \cdot \int_A y \cdot dA.$$

By solving the integrals and taking into account relations (5.1), (5.4), (5.5) and (5.6), we obtain:

$$I_{z_1} = I_z + y_0^2 \cdot A,$$
$$I_{y_1} = I_y + z_0^2 \cdot A, \qquad \textbf{(5.9)}$$
$$I_{z_1 y_1} = I_z + z_0 \cdot y_0 \cdot A.$$

Thus, **the moment of inertia with respect to a parallel axis is equal to the sum of the moment with respect to the parallel central axis and the product of the surface area and the square of the span between the axes**.

The centrifugal moment of inertia with respect to the parallel axes is equal to the sum of the moment of inertia with respect to the own central axes and the product of the area and the coordinates of the center of gravity of the area in the new system. Thus, the value and the sign of the moment of centrifugal inertia are determined by **the sign of the product of the coordinates of the center of gravity of the section in the new system.**

The moments of inertia of a section composed of n simple sections of areas A_i (or A decomposed into n simple sections A_i), related to the system of axes Oyz (usually system of central axes), are calculated with the relations:

$$I_z = \sum_{i=1}^{n} \left(I_{zi} + A_i \cdot y_{0i}^2\right), \quad I_y = \sum_{i=1}^{n} \left(I_{yi} + A_i \cdot z_{0i}^2\right), \quad I_{zy} = \sum_{i=1}^{n} \left(I_{z_i y_i} + A_i \cdot y_{0i} \cdot z_{0i}\right). \qquad \textbf{(5.10)}$$

where $\mathbf{I_{z_i}}$, $\mathbf{I_{y_i}}$, $\mathbf{I_{z_i y_i}}$ are the axial moments of de inertia, namely centrifugal of each area section A_i related to its own central axes ($O_{i1}z_{i1}y_{i1}$), parallel to axes Ozy and z_{oi}, y_{oi}, are the coordinates of the centers of gravity O_i in the system of axes Ozy.

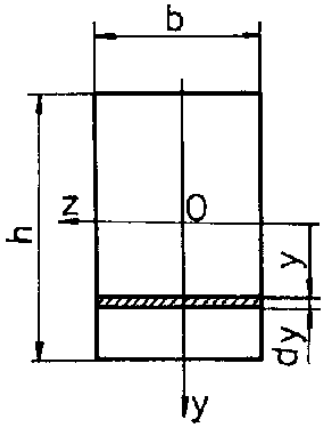


Fig. 5.3

5.5 Exercises

5.5.1 Central moments of inertia of a rectangle (fig.5.3)

Axes Ozy are main central axes of inertia (symmetry axes). The area element dA = b·dy is chosen, on ordinate y. By substituting it in the first relation (5.5), the following is obtained:

$$\mathbf{I_z = \int_A y^2 \cdot dA = \int_{-h/2}^{h/2} y^2 \cdot b \cdot dy = \frac{b}{3} \cdot \left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3 \right] = \frac{b \cdot h^3}{12}} \cdot$$

Doing the same in relation to axis Oy axis, these formulas are obtained:

$$\mathbf{I_z = \frac{b \cdot h^3}{12} \quad , \quad I_y = \frac{b \cdot h^3}{12} \quad , \quad I_{zy} = 0.} \qquad \mathbf{(5.17)}$$

The moment of centrifugal inertia is null because the z and y axes are axes of symmetry.

5.5.2. Central moments of inertia of the circular cross section (fig. 5.4)

A system of main central axes is chosen, having its origin in the center of the circle and the area element dA = 2π· r·dr.

By applying relation (5.7), the polar moment of inertia is obtained:

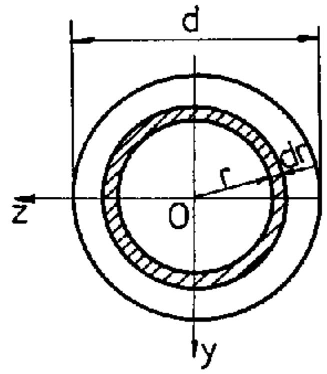


Fig. 5.4

$$\mathbf{I_p = I_0 = \int_A r^2 \cdot dA = 2\pi \cdot \int_0^{d/2} r^3 \cdot dr = \frac{2\pi}{4} \cdot \left(\frac{d}{2}\right)^4}$$

therefore,

$$\mathbf{I_P = \frac{\pi \cdot d^4}{32}}. \qquad \mathbf{(5.18)}$$

Since axes z and y are diametrical (equatorial) axes of the circle, there is the equality $I_z = I_y$ and relation (5.18) leads to:

$$\mathbf{I_z = I_y = \frac{I_P}{2} = \frac{\pi d^4}{64}}, \qquad \mathbf{I_{zy} = 0}. \qquad \mathbf{(5.19)}$$

5.5.3. Annular cross section or the annulus (fig. 5.5)

Considering that this section is composed of a circle of diameter D, from which another circle of diameter d is subtracted, the following moment of polar inertia is obtained:

$$\mathbf{I_P = \frac{D^4}{32} - \frac{d^4}{32} = \frac{\pi \cdot d^4}{32} \cdot \left[1 - \left(\frac{d}{D}\right)^4\right]} \quad \mathbf{(5.20)}$$

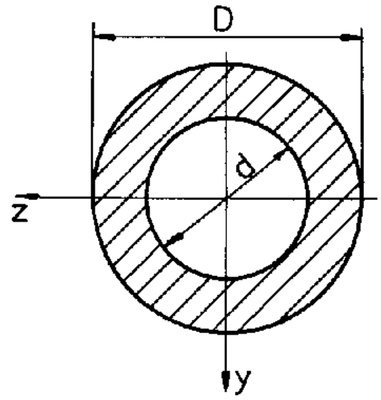


Fig. 5.5

In a similar manner for the axial moments of inertia, we obtain:

$$\mathbf{I_z = I_y = \frac{\pi \cdot D^4}{64} \cdot \left[1 - \left(\frac{d}{D}\right)^4\right]} \qquad \mathbf{(5.21)}$$

Ratio $\mathbf{k = \frac{d}{D}}$ is a constructive factor of the annular section, so that the polar or the axial moments of inertia depend only on the outer diameter D and can be written:

$$\mathbf{I_z = I_y = \frac{\pi \cdot D^4}{64} \cdot \left(1 - k^4\right)} \text{ and } \mathbf{I_p = \frac{\pi \cdot D^4}{32} \cdot \left(1 - k^4\right)}. \qquad \mathbf{(5.21,a)}$$

5.5.4. The section made up of two rectangles with axis Oy as the axis of symmetry (fig.5.6)

a) The position of the center of gravity in the $O_1z_1y_1$ system of axes results:

$$z_G = 0\,, \quad y_G = \frac{6 \cdot 4 \cdot 0 + 2 \cdot 12 \cdot 8}{6 \cdot 4 + 2 \cdot 12} = 4\,cm\,.$$

The main axes of Ozy were plotted in figure 5.9 and the positions of the centroids of the simple sections were marked.

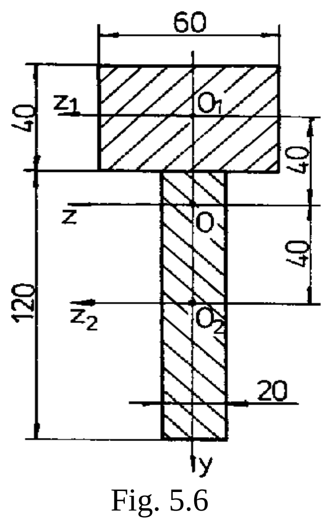


Fig. 5.6

b) The moments of inertia with respect to the central axes are:

$I_{zy} = 0$ (there is an axis of symmetry),

$$I_y = \left(I_{zi} + A_i \cdot z_{oi}^2\right) = \frac{6^3 \cdot 4}{12} + 6 \cdot 4 \cdot 0^2 + \frac{2^3 \cdot 12}{12} + 2 \cdot 12 \cdot 0 = 80\ cm^4$$

$$I_z = \left(I_{yi} + A_i \cdot y_{oi}^2\right) = \frac{6 \cdot 4^3}{12} + 24 \cdot 4^2 + \frac{2 \cdot 12^3}{12} + 24 \cdot 4^3 = 1088\ cm^4$$

5.6. Radius of Gyration

By definition, the following geometric quantities

$$\mathbf{i_z = \sqrt{\frac{I_z}{A}}} \quad \text{and} \quad \mathbf{i_y = \sqrt{\frac{I_y}{A}}}, \qquad \textbf{(5.22)}$$

are called **radii of inertia (gyration).**

The defining relationships (5.22) can be applied to any axial moment of inertia: I_z, I_y, I_u, I_v, I_1, I_2 etc.

5.7. Strength Modulus

When calculating the strength modulus, axes Oz and Oy are considered to be main central axes.

The geometric quantities:

$$W_z = \frac{I_z}{|y_{max}|} \quad \text{and} \quad W_y = \frac{I_y}{|z_{max}|}, \qquad \textbf{(5.23)}$$

are called strength moduli to axis Oz, respectively Oy. In the relations above y_{max}, respectively z_{max} represent: the distance between the farthest point of the section to axis Oz, and respectively to axis Oy.

The following quantity

$$W_P = \frac{I_P}{R_{max}}, \qquad \textbf{(5.24)}$$

is called a **polar strength modulus**. R_{max} is the distance between the center of gravity (the pole of the section) and the point farthest from the pole.

In the case of the rectangular sections, the axial strength moduli are:

$$W_z = \frac{I_z}{|y_{max}|} = \frac{b \cdot h^3}{12} \cdot \frac{2}{h} = \frac{b \cdot h^2}{6},$$
$$W_y = \frac{I_y}{|z_{max}|} = \frac{b^3 \cdot h}{12} \cdot \frac{2}{b} = \frac{b^2 \cdot h}{6}. \qquad \textbf{(5.25)}$$

For the circular section, the axial strength moduli are:

$$W_z = W_y = \frac{I_z}{|y_{max}|} = \frac{\pi \cdot d^4}{64} \cdot \frac{2}{d} = \frac{\pi \cdot d^3}{32}, \qquad \textbf{(5.26)}$$

and the polar strength modulus will be:

$$W_P = \frac{I_P}{R} = \frac{\pi \cdot d^4}{32} \cdot \frac{2}{d} = \frac{\pi \cdot d^3}{16}. \qquad \textbf{(5.27)}$$

In the case of the annular section (fig. 5.8) the following formulas are obtained:

$$W_z = W_y = \frac{\pi \cdot D^3}{32} \cdot \left[1 - \left(\frac{d}{D}\right)\right]^4 = \frac{\pi \cdot D^3}{32} \cdot \left(1 - k^4\right),$$
$$W_p = \frac{\pi \cdot D^3}{16} \cdot \left[1 - \left(\frac{d}{D}\right)\right]^4 = \frac{\pi \cdot D^3}{16}\left(1 - k^4\right). \qquad \textbf{(5.28)}$$

From analyzing formulas (5.28), compared to (5.20) and (5.21), it should be noted that **the strength moduli of the compound sections cannot be obtained by summing the strength moduli of the component figures, but only by applying relations (5.26) and (5.27)**.

6. AXIAL LOADS – S16

6.1. Stresses and Strains - S16

A beam is axially loaded, if the only forces that occur in its cross sections are the axial forces N, which can be constant or variable. The value of the axial force N near a section is equal to the sum of the projections on the axis of the beam of all the forces located to the left or to the right of the considered section.

In order to study the stresses it is recommended to represent the diagram of the axial forces for determining the hazardous section (or sections). The axial forces are considered **positive when they cause the elongation and negative when they cause the compression of the cross section**.

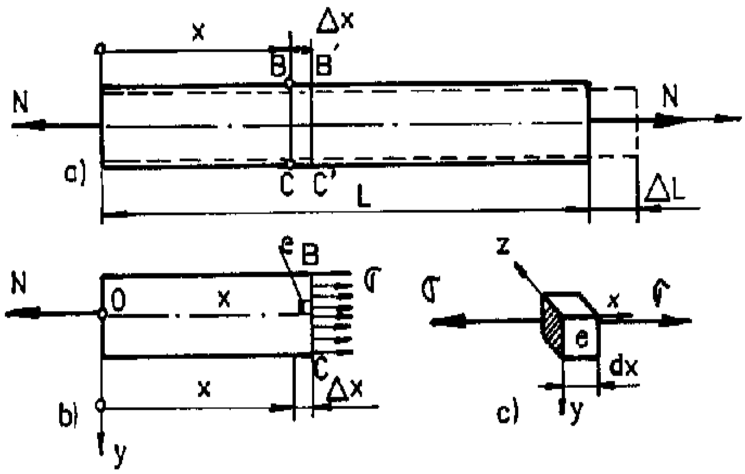


Fig. 6.1

The axial force is the result of all the normal stresses that develop in a certain cross section. In order to determine the normal stresses, we consider an axially loaded beam, of length L, made of a homogeneous and isotropic material and having a constant cross section of area A.

By applying an axial force N, the beam elongates with quantity ΔL. An arbitrary section BC, located at abscissa x, displaces by quantity Δx. According to Bernoulli's hypothesis that a plane and normal section on the axis of the bar before deformation remains plane and normal on the axis of the beam after deformation, it results that all the points of section BC displace axially by the same value Δx= ct. and:

$$\boldsymbol{\varepsilon_x = \frac{\Delta x}{x} = ct.}$$

According to Hooke's law, the constant normal stresses correspond to the constant specific elongation:

$$\boldsymbol{\sigma = E \cdot \varepsilon}.$$

Due to the hypothesis we considered the material to be isotropic, thus the modulus of longitudinal elasticity is constant (E = ct.) and it results σ = ct.

Therefore, **the normal stresses are evenly distributed on the cross-section surface** (fig.6.1, b).

From the equation of balance written for the left side of the beam (fig.6.1, b) it results:

$$N = \int_A \sigma \cdot dA = \sigma \cdot \int_A dA = \sigma \cdot A.$$

From this equation we obtain the value of the normal stress for the tensile or the compression stress:

$$\boldsymbol{\sigma = \frac{N}{A}}. \qquad \textbf{(6.1)}$$

The state of stress, in this case, is a uniaxial state (fig. 6.1, c).

Since the material is considered to obey Hooke's law, the specific deformation for the axial stresses has the following expression:

$$\boldsymbol{\varepsilon = \frac{\sigma}{E} = \frac{N}{E \cdot A}}. \qquad \textbf{(6.2)}$$

The value of the elongation, respectively of the total shortening of the beam is:

$$\boldsymbol{\Delta L = \varepsilon \cdot L = \frac{N \cdot L}{E \cdot A}}. \qquad \textbf{(6.3,a)}$$

If on the length of the beam the quantities N, E, and A are variable, or constant on certain portions of the beam, the elongation is calculated with the relation:

$$\boldsymbol{\Delta L = \int_L \frac{N}{E \cdot A} dx} \quad \text{or} \quad \boldsymbol{\Delta L = \sum \frac{N \cdot L}{E \cdot A}}. \qquad \textbf{(6.3,b)}$$

The elongation (shortening) ΔL is smaller as t**he EA product is bigger** and therefore the EA product **is called the elongation-compression modulus of rigidity**.

The relations solved above and those that will be solved below are valid for both the elongation and the compression loads.

The long beams subjected to compressive stresses should be tested for buckling. The buckling phenomenon (also called loss of elastic stability) occurs before the stresses produced by the compressive stress reach the value σ_a. Therefore, the compression strength can be calculated only for short beams whose length does not exceed 15 times the size of the smallest cross-section:

$$\boldsymbol{L \le 15 \cdot d_{min}}, \qquad \textbf{(6.4)}$$

and the buckling calculation will be done for $\mathbf{L \geq 15 \cdot d_{min}}$.

6.2. Calculation of Tensile - Compressive Strength - S14/17

The relations deduced above are used to solve the problems of strength of materials: testing, loading capacity and dimensioning. These problems are solved by respecting both the strength condition ($\sigma_{max} \leq \sigma_a$) and the rigidity condition ($\varepsilon_{max} \leq \varepsilon_a$ or $\Delta L_{max} \leq \Delta L_a$). The allowable strength is (σ_a) and the allowable deformation is (ε_a, ΔL_a).

Taking these into consideration, the calculation formulas are deduced on the problems:

a) The **testing** of a part subjected to an axial stress consists in determining the maximum stress, respectively the maximum strain and comparing the value obtained with the allowable one. The resulting value must not exceed the allowable value, i.e.:

- from the strength condition:

$$\boldsymbol{\sigma_{ef} = \frac{N_{max}}{A_{ef}} \leq \sigma_a} \qquad \textbf{(6.5)}$$

- from the rigidity condition:

$$\boldsymbol{\varepsilon_{max} = \frac{N}{E \cdot A} \leq \varepsilon_a} \text{ or } \boldsymbol{\Delta L_{max} = \frac{N \cdot L}{E \cdot A} \leq \Delta L_a}. \qquad \textbf{(6.5,a)}$$

In the first relation (6.5) A_{ef} refers to the value of the effective area of the section.

The inequalities in formulas (6.5) are not totally restrictive, meaning that **the boundary values (σ_a, ε_a, ΔL_a) can be exceeded by 3 to 5%.** In order to meet the condition of the efficient use of the beam **it is recommended that the actual value of the stress or strain should not be less than 80% of the allowable value**.

If the beam simultaneously meets the following conditions:

$$\mathbf{0{,}8 \cdot \sigma_a \le \sigma_{max} \le 1{,}05 \cdot \sigma_a},$$

$$\mathbf{0{,}8 \cdot \varepsilon_a \le \varepsilon_{max} \le 1{,}05 \cdot \varepsilon_a}, \qquad \mathbf{(6.5,c)}$$

$$\mathbf{0{,}8 \cdot \Delta L_a \le \Delta L_{max} \le 1{,}05 \cdot \Delta L_a}.$$

we will say that **THE BEAM ENDURES**.

If a single measurement calculated from the relations (6.5) exceeds the allowable value by 5%, then we say that **THE BEAM FAILS**.

If the calculated quantities are below 80% of the allowable values, it is said that **THE BEAM IS OVER-SIZED**.

b) **The maximum applicable load** is calculated for both the beams for which the value of the load is unknown, and for those that were tested and did not correspond to the imposed load, because they were either under-sized and/or over-sized.

Knowing the dimensions of the cross-section of the beam, the material from which it is made (σ_a) and the deformation conditions (ε_a, ΔL_a), the maximum axial force is determined with one of the formulas:

- from the strength condition:

$$\mathbf{N_{cap} = A_{ef} \cdot \sigma_a}, \qquad \mathbf{(6.6,a)}$$

- from the rigidity condition:

$$\mathbf{N_{cap} = E \cdot A_{ef} \cdot \varepsilon_a} \quad \text{or} \quad N_{cap} = \frac{E \cdot A_{ef} \cdot \Delta L_a}{L}. \qquad \mathbf{(6.6,b)}$$

If we take into account both conditions (strength and rigidity), we reach two different values for the maximum load. The lowest value **is taken into consideration.**

The engineer must always choose the value of the force, so that **the beam will withstand, will be efficiently used, and the value of the force will be the estimated one**. Therefore:

$$\mathbf{0{,}8 \cdot P_{cap\,min} \le P \le 1{,}05 \cdot P_{cap\,min}}. \qquad \mathbf{(6.7)}$$

c) **Dimensioning** is the most difficult problem, and it consists in determining the dimensions of the cross section of the beam, so that it can withstand the imposed loads.

The first operation for sizing is to find the maximum normal stress. This results from the axial force diagram. Then, the material for the beam is chosen and the values of the allowable strength and of the allowable strain are selected.

The required area of the cross-section is calculated with the relations:

- from the strength condition:

$$\mathbf{A_{nec} = \frac{N_{max}}{\sigma_a}}, \qquad \textbf{(6.8,a)}$$

- from the rigidity condition:

$$\mathbf{A_{nec} = \frac{N_{max}}{E \cdot \varepsilon_a}} \quad \text{or} \quad A_{nec} = \frac{L \cdot N_{max}}{E \cdot \Delta L_a}. \qquad \textbf{(6.8,b)}$$

As with the load capacity, two different values for the area can be obtained here. This time **the highest value is taken into account to meet both conditions simultaneously**. Also the cross-sectional dimensions of the beams are estimated and **the standardized value must always be chosen**. For this purpose, the shape and the dimensions of the sections given in the tables with standardized profiles **are chosen**.

Exercise 6.1. The beam in figure 6.2 should be dimensioned and the displacement of point 5 should be determined, knowing that σ_a=150 [MPa], ε_a= 250

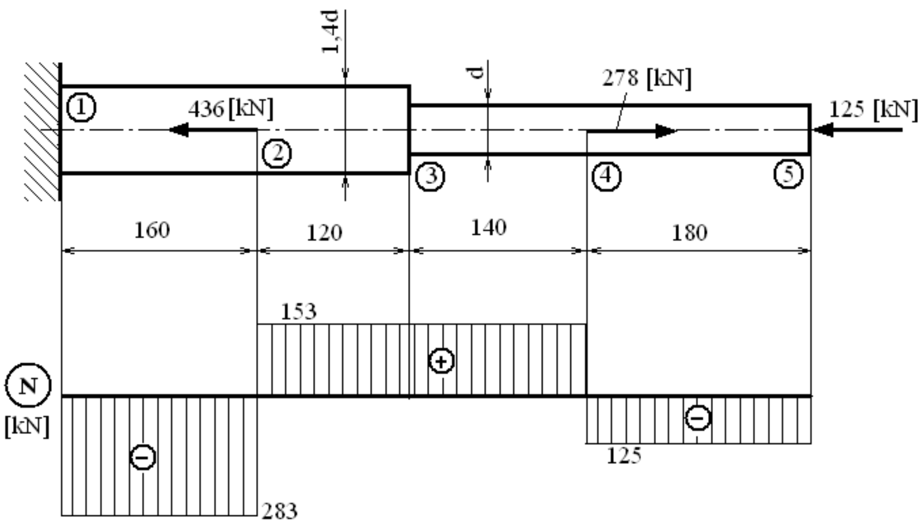


Figure 6.2

[μm/m] and E=210 [GPa].

The hazardous areas are on sections 1-2 and 3-4.

a) The calculation under the strength condition:

- for section 3-4

$$d = \sqrt{\frac{4N}{\pi \cdot \sigma_a}} = \sqrt{\frac{4 \cdot 153 \cdot 10^3}{\pi \cdot 150}} = 36.04\ [mm]$$

- for section 1-2

$$A_{nec} = \frac{\pi \cdot (1{,}4 \cdot d)^2}{4} = \frac{N_{1-2}}{\sigma_a}$$

$$d = \frac{1}{1{,}4} \cdot \sqrt{\frac{4N}{\pi \cdot \sigma_a}} = \frac{1}{1{,}4} \cdot \sqrt{\frac{4 \cdot 283 \cdot 10^3}{\pi \cdot 150}} = 35.01\ [mm]$$

b) The calculation under the rigidity condition:

- for section 3-4

$$\varepsilon_a = 250 [\frac{\mu m}{m}] = 250 \cdot 10^{-6} [\frac{mm}{mm}]$$

$$d = \sqrt{\frac{4N}{\pi \cdot \varepsilon_a \cdot E}} = \sqrt{\frac{4 \cdot 153}{\pi \cdot 250 \cdot 10^{-6} \cdot 210}} = 43.51\ [mm]$$

- for section 1-2

$$A_{nec} = \frac{\pi \cdot (1{,}4 \cdot d)^2}{4} = \frac{N}{\varepsilon \cdot E}$$

$$d = \frac{1}{1{,}4} \cdot \sqrt{\frac{4N}{\pi \cdot \varepsilon_a \cdot E}} = \frac{1}{1{,}4} \cdot \sqrt{\frac{4 \cdot 283}{\pi \cdot 250 \cdot 10^{-6} \cdot 210}} = 59.17\ [mm]$$

Four values have been obtained for diameter d of the beam. The size of the diameter is chosen by taking into account the highest value of diameter d.

We choose d = 60 [mm].

The calculation of the displacement of point 5:

$$\delta_5 = \sum \frac{N_i \cdot l_i}{E \cdot A_i} = \frac{1}{210} \cdot \frac{4}{\pi \cdot 50^2} \left(\frac{-283 \cdot 160}{1.4^2} + \frac{153 \cdot 120}{1,.4^2} + \frac{153 \cdot 140}{1} + \frac{-125 \cdot 180}{1} \right) = -0.036 [mm]$$

Exercise 6.2. Calculate the maximum load that can be withstood by the beam in figure 6.3, knowing that σ_a=120 [MPa], ε_a= 175 [μm/m].

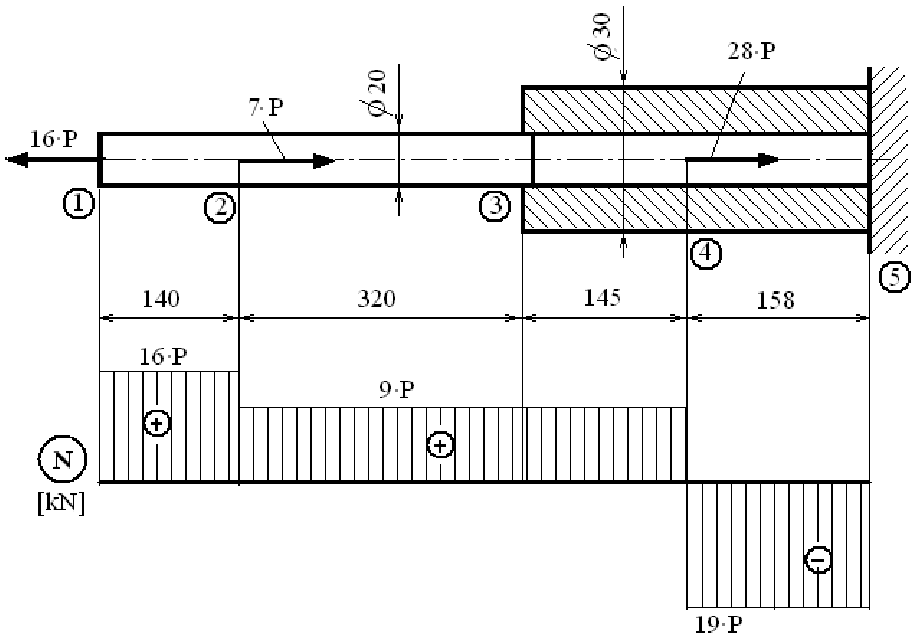


Figure 6.3

The hazardous areas are on sections 1-2 and 4-5.

c) The calculation under the strength condition:

- for section 1-2

$$N_{cap} = 16P = \sigma_a \cdot A_{ef}$$

$$P = \frac{\sigma_a \cdot A_{ef}}{16} = 120 \cdot \frac{\pi \cdot 20^2}{4 \cdot 16} = 2356.2[N]$$

- for section 4-5

$$A_{ef} = \frac{\pi\left(30^2 - 20^2\right)}{4}; \qquad N_{cap} = 19P$$

$$P = \frac{\sigma_a \cdot A_{ef}}{19} = 120 \cdot \frac{\pi\left(30^2 - 20^2\right)}{4 \cdot 19} = 2480.2[N]$$

b) The calculation under the rigidity condition:

- for section 1-2

$$\varepsilon_a = 175[\frac{\mu m}{m}] = 175 \cdot 10^{-6} [\frac{mm}{mm}]$$

$$N_{cap} = 16 \cdot P = A_{ef} \cdot \varepsilon_a \cdot E$$

$$P = \frac{A_{ef} \cdot \varepsilon_a \cdot E}{16} = \frac{\pi \cdot 20^2 \cdot 175 \cdot 10^{-6} \cdot 210}{4 \cdot 16} = 0.7216[kN]$$

- for section 4-5

$$N_{cap} = 19 \cdot P = A_{ef} \cdot \varepsilon_a \cdot E$$

$$P = \frac{A_{ef} \cdot \varepsilon_a \cdot E}{19} = \frac{\pi \cdot \left(30^2 - 20^2\right) \cdot 175 \cdot 10^{-6} \cdot 210}{4 \cdot 19} = 0.75956[kN]$$

Four values were obtained for the maximum force. The lowest value is chosen. We choose P= 720 [N].

6.3. Statically Indeterminate Beam Systems – S15/18

6.3.1. General notions

The stresses and the strains of a statically determinate beam have been analyzed so far. In engineering practice there are assemblies and subassemblies made of beam systems that can be statically determinate or statically indeterminate.

When the number of the unknown values (reactions and/or stresses) exceeds the number of the static equilibrium equations, the system is called **statically indeterminate**. The difference between the number of the unknown values and the number of the static equations is **the degree of indeterminacy of the system**. To solve this case, **a number of deformation equations equal to the degree of indeterminacy of the system is added to the static equations**. These additional equations are obtained from analyzing the geometric aspect and the compatibility of the beam system.

The axially loaded statically indeterminate systems can be caused by:

- connections that prevent the strain caused by loads or by alterations of the beam temperatures;
- technological or assembling stresses that occur in the beam systems;
- the use in building a beam of several materials with different physical and mechanical characteristics.

In all these situations, the statically indeterminate problems can be solved by covering the following three aspects:

a) **the static aspect** – by writing the static equilibrium equations, the unknown values of the system and the degree of indeterminacy are established;

b) **the geometrical aspect** – by writing a number of deformation equations equal to the degree of indeterminacy;

c) **the physical aspect** – by expressing the deformations mentioned at point b) according to the forces or stresses in the beam.

Thus, after covering the three aspects, the necessary equations are obtained from the static and the physical aspects. By solving these equations, the solutions of the statically indeterminate problem are expressed in forces, stresses or strains.

6.3.2. Beams with deformations prevented by bearings

Exercise 6.3. The cantilever (or hinged) beam, fixed at the two ends (fig. 6.4).

We consider the straight, prismatic beam fixed or hinged at the two ends and loaded with the axial load P at a point at distance **a** from bearing 1 (respectively at distance b from bearing 2).

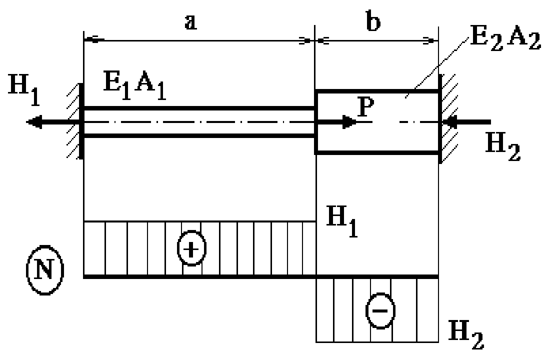


Fig. 6.4

Solution: The reactions in the two bearings are H_1 and H_2.

The static aspect:

$H_1 + H_2 = P$ (simple statically indeterminate system):

The geometric aspect:

$\Delta a + \Delta b = 0$ (the total deformation of the beam must be zero):

The physical aspect:

$$\frac{H_1 \cdot a}{E_1 \cdot A_1} + \frac{H_1 - P}{E_2 \cdot A_2} \cdot b = 0,$$

which leads to:

$$H_1 = \frac{P}{1 + \frac{a}{b} \cdot \frac{E_2}{E_1} \cdot \frac{A_2}{A}} \qquad \text{and then} \qquad H_2 = P - H_1.$$

Knowing the values of the reactions, the axial forces variation diagram can be drawn, and then the resistance calculation can be performed.

If $E_1 \cdot A_1 = E_2 \cdot A_2 = E \cdot A$ and $L = a + b$, then;

$$H_1 = \frac{b}{L} \cdot P \quad \text{and} \quad H_2 = \frac{a}{L} \cdot P.$$

6.3.3 Stresses caused by arrested expansions

A straight beam of length L, which can expand freely, when it is uniformly heating, elongates by (fig. 6.5, a):

$$\mathbf{\Delta L = \alpha \cdot L \cdot \Delta t}$$

where α is the coefficient of linear expansion and $\mathbf{\Delta t} = t - \mathbf{t_0}$ is the increase in temperature. The specific elongation of the beam is:

$$\boldsymbol{\varepsilon = \frac{\Delta L}{L} = \alpha \cdot \Delta t}$$

When the bar has static hinges or is fixed at both ends (Fig. 6.5, b), which arrests the expansion, an axial compressive force occurs in the beam. This force should shorten the bar with the elongation caused by the increase in temperature (fig. 6.5, c and d), that is with:

$$\mathbf{\Delta L = \alpha \cdot \Delta t \cdot L = \frac{N \cdot L}{E \cdot A}},$$

from which the formula for the axial compressive force is obtained:

$$\mathbf{N = \alpha \cdot E \cdot A \cdot \Delta t}$$

As such, the following stress will occur in the beam:

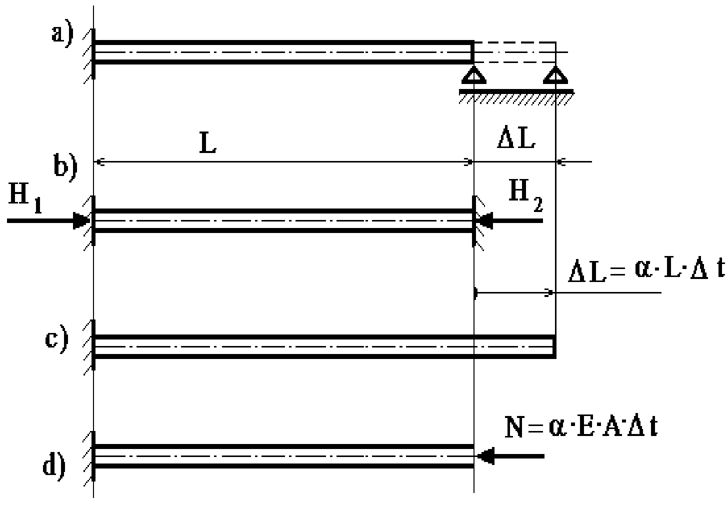


Fig. 6.5

$$\boldsymbol{\sigma = \frac{N}{A} = \alpha \cdot E \cdot \Delta t.}$$

the stresses in the arrested expansion beams.

Exercise 6.4. Section I20 (A = 35.5 cm^2, E=210 GPa, α=12×10^{-6} °C^{-1}) mounted as in Figure 6.6, at 5°C temperature, leaving an expansion space a = 2 mm. Determine the stress and the strain in the beam at a temperature of 45°C.

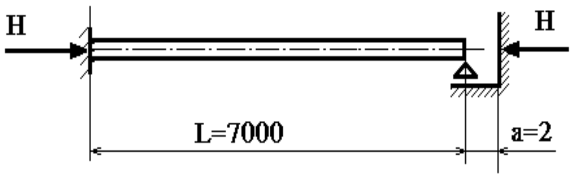


Fig. 6.6

Solution: The space required for the arrested expansion is:

$\Delta L=\alpha \cdot \Delta t \cdot L=12 \cdot 10^{-6} \cdot 40 \cdot 7000 = 3.36$ mm.

Since ΔL = 3.36 mm > a = 2 mm, the expansion is arrested. Therefore, the system is statically indeterminate. The equations of the three aspects are the following:

a) $N = H_1 = H_2,$

b) $(\Delta L)_T - (\Delta L)_N = a,$

c) $\alpha \cdot L \cdot \Delta t - \dfrac{N \cdot L}{E \cdot A} = a.$

which leads to:

$$H = N = \left(\alpha \cdot \Delta t - \frac{a}{L}\right) \cdot E \cdot A = \left(12 \cdot 10^{-6} \cdot 40 - \frac{2}{7000}\right) \cdot 210 \cdot 3350 = 136.7\,kN,$$

$$\sigma_{ef} = \frac{N}{A} = \frac{136{,}7 \cdot 10^3}{3350} = 40{,}8\,MPa < \sigma_a = 150\,MPa.$$

Note: The stresses and the strains that occur due to the arrested expansion are supplementary and are added to those produced by the live loads.

7. TORSION OF STRAIGHTS BEAMS – S16

7.1. General Notions

A beam is subjected to torsion the stress in any cross section of the beam is a **moment of torsion** (torque).

The torque in a given section is equal to the sum of all the moments of the forces to the left or to the right of the considered section.

$$M_t = \sum \left(P_i \cdot R_i + M_{xi} \right) \qquad \textbf{(7.1)}$$

where P_i are the external forces normal on the axis of the beam, R_i the distances from the axis to the bearings of the forces, and M_{xi} are the external moments oriented in the direction of axis Ox.

If the beam transmits a power *P**, in [kW], at speed *n*, in rotations per minute, then the value of the torque is:

$$M_t = \frac{30}{\pi} \cdot \frac{P^*}{n} \qquad \textbf{[kNm]} \qquad \textbf{(7.2)}$$

When the value of the torque varies along the beam, in order to calculate strength, it is recommended to draw the torque diagrams and to specify the hazardous section (or sections).

In the field of activity of the mechanical engineer there are frequent applications of torsion of straight beams, such as: shafts, driving axles, screws, etc.

7.2. Stresses and Strains Occurring in Circular and Annular Cross Section Beams Subjected to Torsion

We consider a straight beam of circular and constant longitudinal section made of a continuous, homogeneous, isotropic material that obeys Hooke's law. Circles and generators are drawn on the surface of the beam, which form a network of curvilinear rectangles, of which the elementary rectangle ABCD is considered. We take into consideration section (1) located at distance dx from section (2), (fig.7.1, a).

After applying the torque, the beam deforms as shown in figure (7.1, b). the analysis of the deformation of the beam proves that:

a) the circles in the cross section planes remain circles contained in the same cross section planes, and the distance between them does not change significantly (Bernoulli's hypothesis is confirmed, for the points on the outer surface and also extends to the points inside the bar);

b) the rectangular elements on the lateral surface are transformed into parallelograms whose sides maintain their length;

c) the two generators (fibers) remain parallel to each other, but change to helical lines.

Thus, any rectangular element on the surface of the beam is deformed by pure sliding into a parallelogram (fig.7.1,c).

$$\boldsymbol{\gamma_0 = \lim_{\Delta x \to 0} \frac{\Delta e}{\Delta x} = \frac{de}{dx}}.$$

Arc Δe is the displacement through sliding of point A or B in A’ and B’, respectively. Thus, circle (1) rotates by arc Δe = AA’= BB’ against circle (2). The angle at which section (1) rotates related to section (2), which is at distance dx from section (1), is called **angular deformation or relative rotation** and is marked by dφ (fig. 7.2). It can be written as:

$$\boldsymbol{\Delta e = AA' = BB' = R \cdot d\varphi = \gamma_0 \cdot dx.}$$

It results:

$$\boldsymbol{\gamma_0 = R \cdot \frac{d\varphi}{dx} = R \cdot \theta,}$$

where quantity:

$$\boldsymbol{\theta = \frac{d\varphi}{dx}}, \qquad \textbf{(7.3)}$$

is called **specific rotation.**

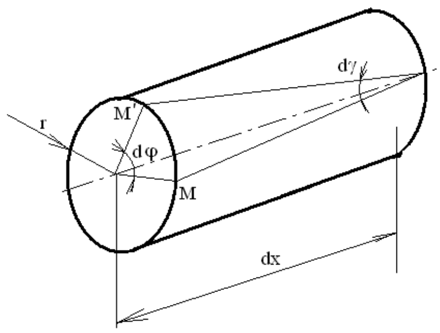


Fig. 7.2

Similarly, for arc MMă, located at distance r from the axis of the beam, the following is obtained:

$$\mathbf{MM' = r \cdot d\varphi = \gamma \cdot dx},$$

from which the specific sliding at radius r is calculated.

$$\boldsymbol{\gamma = r \cdot \frac{d\varphi}{dx} = r \cdot \theta.} \qquad \textbf{(7.4)}$$

Since the material of the beam is considered continuous, homogeneous, isotropic and elastic, the elemental rotation $d\varphi$ has the same value for all the points of a section. Since $d\varphi$ is constant throughout the entire cross section and **the specific rotation θ** remains constant too throughout the entire length dx. Thus from relation (7.4) it follows that the **specific sliding varies linearly according to r**. It has a null value on the axis of the beam and a maximum value ($\gamma_0 = R \cdot \theta$) on the outer contour. Due to the sliding deformations, tangential stresses occur in the beam, which can be determined, for loads in the linear-elastic field, by means of Hooke's law:

$$\boldsymbol{\tau = G \cdot \gamma = G \cdot r \cdot \theta}. \qquad \textbf{(7.5)}$$

We consider an area element dA at distance R = d/2 (therefore on the outer contour of the beam, (fig. 7.2) acted upon by a tangential stress τ having an arbitrary direction. It has the following components: τ_{xs}- tangent to the contour and τ_{sx} the radial. According to the duality of the tangential stresses, a stress τ_{sx} on the outer surface of the beam will correspond to a stress τ_{xs}. Because no axial shearing forces have been taken into account on the outer surface of the beam which will produce stress τ_{sx}, this is null.

Thus, **the tangential stresses contained in the cross section are perpendicular to the radius and vary proportionally with it**. According to the law of the duality of tangential stresses, the pair of stress τ_{xs} is stress τ_{sx} and is contained in the axial plane (fig.7.2), that is:

$$\boldsymbol{\tau = \tau_{xs} = \tau_{sx} = \theta \cdot G \cdot r.} \qquad \textbf{(7.5, a)}$$

By writing the equation of equivalence between the effort M_t and the stresses in the plane of the cross section, we will obtain:

$$\mathbf{M_t = \int_A r \cdot \left(\tau \cdot dA\right)}$$

and by replacing τ in expression (7.5), the following is obtained:

$$\mathbf{M_t = \theta \cdot G \cdot \int_A r^2 \cdot dA = \theta \cdot G \cdot I_p}. \qquad \mathbf{(7.6)}$$

In the above relations, we took into account that:

$$\mathbf{\int_A r^2 \cdot dA = I_p},$$

is the moment of polar inertia)

By substituting the quantities θ · G from (7.6) with the expression resulting from (7.5), we obtain the formula of the tangential stress at circular cross section beams subjected to torsion:

$$\mathbf{\tau = \frac{M_t}{I_p} \cdot r,} \qquad \mathbf{(7.7)}$$

which shows that the tangential stress varies linearly depending on the radius.

From relation (7.7), which is graphically represented in figure (7.2), it results that the tangential stresses are maximum on the outer contour of the beam:

$$\mathbf{\tau_{max} = \frac{M_t}{I_p} \cdot R = \frac{M_t}{W_p}}, \qquad \mathbf{(7.8)}$$

where Wp is the polar strength modulus and is given by the relation:

$$\mathbf{W_p = \frac{I_p}{R_{max}}}. \qquad \mathbf{(7.9)}$$

The formula for the specific rotation results from expression (7.6) and is the following:

$$\mathbf{\theta = \frac{M_t}{G \cdot I_p}}. \qquad \mathbf{(7.10)}$$

Therefore, the specific rotation is directly proportional to the moment of torsion and inversely proportional to the G·I_P product, which is called torsion rigidity of the circular and annular cross section bars. The specific rotation is measured in rad/m, or degrees/m.

The angular deformation of a beam of length L or the relative rotation of the beam, marked with Δφ, which represents the angle by which the final section rotates related to the initial one, is obtained from relation (7.3) and (7.10), as follows:

$$\Delta\varphi = \int_L d\varphi = \int_L \theta \cdot dx = \int_L \frac{M_t \cdot dx}{G \cdot I_p}. \qquad \textbf{(7.11)}$$

If the beam is homogeneous, of constant section and stress Mt is constant throughout the length L, by integrating the relation (7.11), the following is obtained:

$$\Delta\varphi = \frac{M_t \cdot L}{G \cdot I_p} \qquad \textbf{(7.11,a)}$$

and if the values of the quantities below the integral (7.11) are constant on sections of the length of the bar, then relation (7.11) becomes:

$$\Delta\varphi = \sum \frac{M_{ti} \cdot l_i}{G \cdot I_{pi}}. \qquad \textbf{(7.11, b)}$$

Although relations (7.7), (7.8), (7.10) and (7.11) have been calculated for the circular section, they can be similarly solved for the annular section as well.

In formulas (7.6) to (7.11), the quantities I_p and W_p are mentioned and have the following expressions:

$$I_p = \frac{\pi \cdot d^4}{32}, \qquad W_p = \frac{\pi \cdot d^3}{16}, \qquad \textbf{(7.12, a)}$$

for the circular cross section and:

$$I_p = \frac{\pi \cdot D^4}{32} \cdot \left(1 - k^4\right), \qquad W_p = \frac{\pi \cdot D^3}{16} \cdot \left(1 - k^4\right) \qquad \textbf{(7.12, b)}$$

for the annular cross section, where $k = \dfrac{d}{D}$.

7.3. Calculation of the Torsional Strength of Circular Cross Section Beams

The calculation of torsional strength involves solving the problems of verification, maximum applicable load and dimensioning. This calculation is based on the traditional established formula of the strength condition:

$$\tau_{max} \le \tau_a\,, \qquad (7.13)$$

as well as the rigidity condition:

$$\theta_{max} \le \theta_a \quad \text{or} \quad \Delta\varphi_{max} \le \Delta\varphi\,, \qquad (7.14)$$

where τ_{max} is obtained with formula (7.8), θ_{max} with formula (7.10) and $\Delta\varphi$ with one of the formulas (7.11).

The values of the allowable torsional strength τ_a, respectively θ_a or $\Delta\varphi_a$ are established for each MM depending on the material, the operating conditions, the functional role, the mode of considering the forces, etc.

1. The verification problem is solved using the formulas:

$$\tau_{max} = \frac{M_t}{W_p} \le \tau_a \qquad (7.15,\ a)$$

$$\theta_{max} = \frac{M_t}{G \cdot I_p} \le \theta_a \quad \text{or} \quad \Delta\varphi_{max} = \frac{M_t}{G \cdot I_p} \le \Delta\varphi \qquad (7.15,\ b)$$

Depending on the results obtained, the following verdicts will be given:

a) THE BEAM ENDURES, if all the calculated values (τ, θ, or $\Delta\varphi$) are below the allowable quantity and at least one exceeds 0.8 % of the allowable value;

b) THE BEAM FAILS, if at least one of the values exceeds 5% of the allowable value;

c) THE BEAM IS OVERSIZED, if all the determined values are below 0.8% of the allowable one.

The maximum load is calculated in situations b and c.

2. Maximum applicable load problems are solved with these relations:

$$M_{t,cap} = W_p \cdot \tau_a\,, \qquad (7.16,\ a)$$

$$\mathbf{M_{t,cap} = G \cdot I_p \cdot \theta_a} \text{ or } \mathbf{M_{t,cap} = \frac{G \cdot I_p \cdot \Delta\varphi_a}{L}}. \qquad \textbf{(7.16, b)}$$

The value taken into account is the smallest; it will continue to be used to adopt a rounded integer value that meets the condition:

$$\mathbf{0{,}8 \cdot M_{t,cap} < M_t < 1{,}05 \cdot M_{t,cap}}.$$

3. **Solving the dimensioning problems** involves first determining moment M_{tmax} (from the moment diagram), then choosing the material and adopting, τ_a and respectively θ_a or $\Delta\varphi_a$, and for the circular section the following formula is obtained from relations (7.8) and (7.12, a):

$$\mathbf{d_{nec} = \sqrt[3]{\frac{16 \cdot M_{t\,max}}{\pi \cdot \tau_a}}}, \qquad \textbf{(7.17,a)}$$

and from formulas (7.10), (7.11), (7.12, a), for the rigidity condition, these formulas are obtained:

$$d_{nec} = \sqrt[4]{\frac{32M_{t\,max}}{\pi \cdot G \cdot \theta_a}} \text{ or } d_{nec} = \sqrt[4]{\frac{32M_t \cdot L}{\pi \cdot G \cdot \Delta\varphi_a}}. \qquad \textbf{(7.17, b)}$$

In the case of the annular cross section beams, the ratio k = D/d is adopted and from relations (7.8), (7.10), (7.11), (7.12, b), we obtain:

$$D_{nec} = \sqrt[4]{\frac{16M_{t\,max}}{\pi \cdot \tau_a \cdot \left(1-k^4\right)}}, \qquad \textbf{(7.18, a)}$$

and respectively:

$$D_{nec} = \sqrt[4]{\frac{32M_{t\,max}}{\pi \cdot G \cdot \theta_a \cdot \left(1-k^4\right)}} \text{ or } D_{nec} = \sqrt[4]{\frac{32M_{t\,max} \cdot L}{\pi \cdot G \cdot \Delta\varphi_a \cdot \left(1-k^4\right)}}. \qquad \textbf{(7.18, b)}$$

When both the strength and the rigidity conditions are taken into account, two values are obtained for the diameter of the MM. The highest value is adopted in round figures.

Exercise 7.1. Establish the dimension of a steel shaft (G = $8.1 \cdot 10^3$ MPa, τ_a = 80 MPa, θ_a = 1 degree/m) that transmits a power of P^*= 30 kW at a velocity n = 200 rot/min. The shaft will be calculated in the two cases:

a) circular cross section

b)annular cross section k= D/d = 0,7.

Solution:

The torque moment is determined as being:

$$M_t = \frac{30}{\pi} \cdot \frac{P^*}{n} = \frac{30 \cdot 30}{\pi \cdot 200} = 1.432\,\text{kNm}$$

a) The circular cross-section:

$$d^{\acute{A}} = \sqrt[3]{\frac{16 M_t}{\pi \cdot \tau_a}} = \sqrt[3]{\frac{16 \cdot 1432 \cdot 10^3}{\pi \cdot 80}} = 45.01\,\text{mm}\,,$$

$$d''_{nec} = \sqrt[4]{\frac{32 M_t}{\pi \cdot G \cdot \theta_a}} = \sqrt[3]{\frac{32 \cdot 1432 \cdot 10^3}{\pi \cdot 81 \cdot 10^3} \cdot \frac{10^3 \cdot 180}{\pi}} = 56.67\,\text{mm}\,.$$

We choose d = 60 mm.

Note: A lower value (d=55 mm) than the one calculated cannot be selected because, when tested, under the rigidity condition, the following is obtained:

$$\theta_{\max} = \frac{32 M_t}{\pi \cdot G \cdot D^4} = \frac{32 \cdot 1.432 \cdot 10^6}{\pi \cdot 81 \cdot 10^3 \cdot 55^4} \cdot \frac{180 \cdot 10^3}{\pi} = 1.128^o / m > 1.05 \cdot \theta_a.$$

b) The annular cross section:

$$D = \sqrt[3]{\frac{16 M_t}{\pi \cdot \tau_a \cdot \left(1 - k^4\right)}} = \sqrt[3]{\frac{16 \cdot 1.432 \cdot 10^6}{\pi \cdot \left(1 - 0.8^4\right) \cdot 80}} = 53.65\,\text{mm}$$

$$D = \sqrt[4]{\frac{32 M_t}{\pi \cdot G \cdot \theta_a \cdot \left(1 - k^4\right)}} = \sqrt[4]{\frac{32 \cdot 1.432 \cdot 10^6}{\pi \cdot \left(1 - 0{,}8^4\right) \cdot 81 \cdot 10^3} \cdot \frac{10^3 \cdot 180}{3 \cdot \pi}} = 64.66\,\text{mm}$$

We consider: D = 65 mm, d = 52 mm.

The material saved by using this annular section is:

$$\frac{A_I - A_{II}}{A_I} \cdot 100 = \frac{60^2 - \left(65^2 - 52^2\right)}{60^2} \cdot 100 = 57.75\%.$$

7.4. Strain Energy in Circular and Annular Cross Section Beams Subjected to Torsion

Considering a volume element from the beam, under the action of the tangential stresses τ and the specific elementary sliding γ, the specific elementary mechanical work is produced:

$$dL_1 = \tau \cdot d\gamma \,.$$

The load being in the linear elastic domain **τ = G·γ**, therefore $d\gamma = \frac{d\tau}{G}$, and the elementary mechanical work will be equal to the deformation energy, according to the hypothesis that in the elastic domain the whole mechanical work carried out by loading the beam accumulates in its volume as strain energy:

$$dL_1 = dU_1 = \tau \cdot d\gamma = \frac{\tau}{G} \cdot d\tau.$$

The specific deformation energy stored in the unitary volume element when the stress increases slowly from 0 to τwill have the following form:

$$U_1 = \int_0^\tau dU_1 = \int_0^\tau \tau \cdot \frac{d\tau}{G} = \frac{\tau^2}{2G} \qquad \textbf{(7.19)}$$

and that accumulated in the elementary volume is:

$$dU = U_1 \cdot dV = \frac{\tau^2}{2G} \cdot dV.$$

For the circular cross section straight beam, the following applies:

$$\tau = \frac{M_t}{I_p} \cdot r,\;\; I_p = \int_A r^2 \cdot dA = \frac{\pi \cdot d^4}{32},\;\; dV = dA \cdot dx,$$

so the deformation energy accumulated in the circular cross section beam, of length L, subjected to torsion will have the value:

$$U = \int_V dU = \int_V \frac{\tau^2}{2G} \cdot dV = \int_L \frac{M_t^2 \cdot dx}{2G \cdot I_p^2} \int_A r^2 \cdot dA = \int_L \frac{M_t^2 \cdot dx}{2G \cdot I_p}. \qquad \textbf{(7.20)}$$

If the bar is homogeneous, of constant circular cross section and loaded throughout its length by the same M_t, then the accumulated energy of deformation will have the value:

$$U = \frac{M_t^2 \cdot L}{2G \cdot I_p} = \frac{16 \cdot M_t^2 \cdot L}{G \cdot d^4}. \qquad \textbf{(7.21)}$$

If the beam has an annular cross section with the dimensional factor $k = \frac{d}{D}$, the strain energy will be written as:

$$U = \frac{16 M_t^2 \cdot L}{\pi \cdot G \cdot D^4 \cdot \left(1 - k^4\right)}. \qquad \textbf{(7.21,a)}$$

7.5. Calculation of Closed Coil Helical Springs

The helical spring (fig. 7.3) is made of a steel wire having the diameter d which is wound on a cylinder in the form of a spiral. The distance D/2 from the axis of the cylinder to the axis of the wound wire is called **the winding radius**. A force P acts on the spring along the axis of the cylinder. If the force decreases in the center of gravity of a coil, a force P and a moment M = P · R will be obtained.

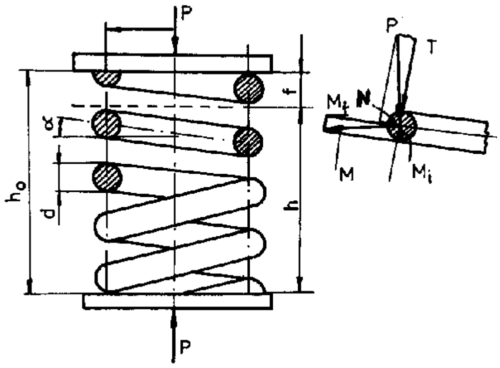


Fig.7.3

By decomposing force P and moment M along the axis of the coil and perpendicular to it, the following stresses are obtained:

$$N = P \cdot sin\alpha ;$$
$$T = P \cdot cos\alpha ;$$
$$M_t = P \cdot R_t \cdot cos\alpha ;$$
$$M_i = P \cdot R \cdot sin\alpha .$$

At closed coil helical springs the winding angle of the coil reaches low values, so that the following approximation can be made:

$sin\alpha \approx 0 \ ; cos\alpha \approx 1$ In this case the stresses in any section of the arc are:

$$M_t = P \cdot R = P \cdot \frac{D}{2} \text{ and } \quad T = P. \qquad \textbf{(7.22)}$$

The tangential stress caused by the shearing force is very small compared to that caused by torque, so that only the effect of the torque will be taken into account. It will turn out:

$$\tau_{max} = \frac{M_t}{W_p} = \frac{16P \cdot D}{2\pi \cdot d^3} == \frac{8 \cdot P \cdot D}{\pi \cdot d^3}. \qquad \textbf{(7.23)}$$

Relation (7.23) is used for calculating the strength for:

verification, maximum applicable load, dimensioning.

The diameter of the coil is obtained from this relation:

$$d_{nec} = \sqrt[3]{\frac{8 \cdot P \cdot D}{\tau_a \cdot \pi}} \qquad \textbf{(7.24)}$$

The allowable strength of the spring steel (OLC55A, OLC65A, OLC75A, OLC85A, 51SI17A, 60SI15A, 51CR11A) is considered: τ_a= 400 to 800 MPa.

The **arc strain** is defined as its compression or elongation under the action of a load (fig.7.5) and is called **sag.**

The relation to determine the sag is obtained by considering the equality between the mechanical work of the applied external forces and the potential strain energy accumulated in the volume of the arc. Taking into account that $L = \frac{P \cdot f}{2}$, and the strain energy is given by relation (7.20), where the following substitutions are made:

$$M_t = \frac{P \cdot D}{2}; \qquad L = \pi \cdot D,$$

the equality L = U becomes:

$$\frac{P \cdot f}{2} = \int_L \frac{M_t^2 \cdot dx}{2G \cdot I_p},$$

respectively:

$$\frac{P \cdot f}{2} = \frac{16 \cdot \pi \cdot D \cdot n \cdot \left(\frac{P \cdot D}{2}\right)^2}{\pi \cdot G \cdot d^4} \quad ,$$

which results in the formula for the sag:

$$f = \frac{8 \cdot P \cdot D^3 \cdot n}{G \cdot d^4}. \qquad \textbf{(7.25)}$$

8. BENDING STAIGHT BEAMS – S17

8.1. Introduction – S17

A beam is acted upon by a bending stress, when there are bending moments in its section. In most cases, the bending stress is caused by traverse forces (acting perpendicularly on the axis of the beam). In these cases, both bending moments and shear forces occur in the cross-sections.

Within this chapter it is admitted that each force passes through the center of gravity of the beam's cross section (the longitudinal axis of the beam) and does not cause an additional torsion load.

Depending on **the nature of the internal stresses** that appear in the beam, the load may be:

- **pure bending,** fig. 8.1 a, when there are only bending moments in the beam's cross section;
- **simple bending**, fig. 8.1 b, when there are both bending moments and shear forces in the beam's cross section.

Depending **on the position of the traverse forces in space**, the bending load may be:

- **oblique pure bending,** fig. 8.1 c, when all the applied forces belong to a single longitudinal central plane, different from the main central planes of inertia;
- **oblique simple bending,** fig. 8.1 d, when all the applied forces belong to a single longitudinal central plane, in which a concentrated force will occur;
- **twisted bending,** when the applied forces are arranged in two or more central planes, (and the direction of the moment vector varies along the beam).

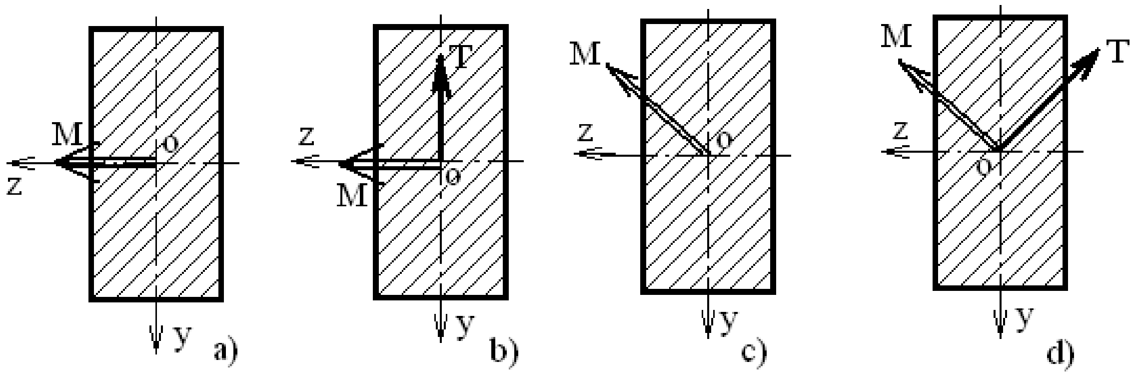


Fig. 8.1

8.2. Stresses and Strains in Straight Beams Subjected to Plane Pure Bending – S18

We consider a straight beam whose cross-section is symmetrical with respect to the vertical plane x0y, subjected to pure bending, by a bending moment directed along the axis 0z (fig.8.1, a).

The beam is made of continuous, homogeneous and isotropic material, having a linear - elastic property (the strains are elastic and proportional to the stresses). By

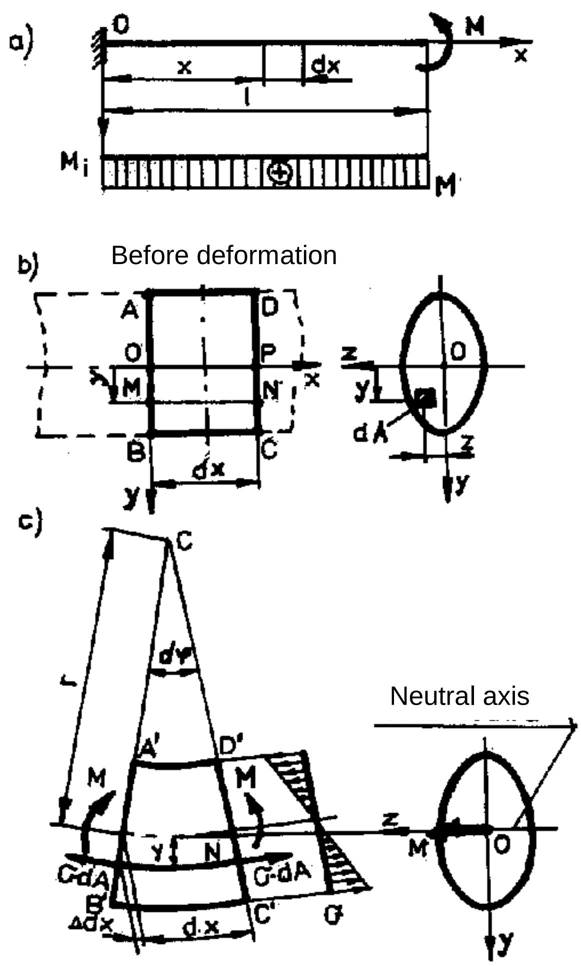


Fig. 8.2

deformation, after applying the bending moment, the plane and normal sections on the axis of the beam before deformation, will be plane and normal on the axis of the beam after deformation too. It is also admitted that all the applied loads are contained in a main central plane of inertia (plane x0y).

From the considered beam, a member of length dx is detached (fig.8.2b). Before applying the bending moment, the fibers of the member AD, BC, MN, are straight and parallel to the axis of the beam 0x. The sections at the ends of the member (AB, CD) are plane and perpendicular to the axis of the beam. Upon loading (the bending moment M is applied), the beam will deform (fig.8.2.c), so that the fibers of the member become curved, and the sections AB and CD will rotate one related to the other at angle dφ. As a result of the deformation, only certain fibers will retain their initial length. These fibers are called **neutral fibers** and form a **neutral surface**. The surface is considered to be plane and is called a **neutral plane**. When M › 0, the upper fibers of the plane compress and the lower fibers extend. The intersection line of the neutral plane with a longitudinal vertical plane (x0y), which contains the axis of the beam, are called **neutral fiber, neutral axis,** or **medium fiber**.

An arbitrary fiber, MN, located on the y coordinate of the neutral plane, before deformation has the length **dx = MN = OP = r ·dφ.**

From this relation the rotation of the section is defined:

$$\omega = \frac{d\varphi}{dx} = \frac{1}{r}.$$

After the beam is deformed, fiber **MN = dx**, will have the following length:

dx + Δdx = M`N` = (r+y) · dφ, and the elongation will be: **Δdx = y · dφ.**

It results that the specific length is:

$$\varepsilon = \frac{\Delta \cdot ds}{ds} = \frac{M^{'}N^{'} - MN}{MN} = \frac{(r+y)\cdot d\varphi - r\cdot d\varphi}{r\cdot d\varphi} = \frac{y}{r}. \qquad \textbf{(8.1)}$$

The normal stress σ, which occurs in the section, on the ordinate y, (next to fiber MN), according to Hooke’s law, will be:

$$\sigma = \varepsilon \cdot E = E \cdot \frac{y}{r}. \qquad \textbf{(8.2)}$$

To obtain the relation between the bending moment and the stresses occurred on the surface of the cross section, **the equivalence equations** are written. In this particular case, when all the elemental forces **σ·dA** are parallel to one another and normal on the cross-section surface, these equations are:

$$\int_{(A)} \sigma \cdot dA = 0, \qquad \int_{(A)} \sigma \cdot z \cdot dA = 0, \quad \int_{(A)} \sigma \cdot y \cdot dA = M. \qquad \textbf{(8.3)}$$

If the expression (9.2) is taken into account, they become:

$$\int_{(A)} y \cdot dA = 0, \qquad \int_{(A)} y \cdot z \cdot dA = 0, \quad \frac{E}{r} \cdot \int_{(A)} y^2 \cdot dA = M. \qquad \textbf{(8.4)}$$

From the obtained relations, one can note the following:

- since:

$$\int_{(A)} y \cdot dA = 0,$$

the neutral axis passes through the center of gravity of the cross section, because only related to a central axis is the static moment of a surface zero. Thus, the origin of the reference system coincides with the center of gravity of the cross section;

From:

$$\int_{(A)} y \cdot z \cdot dA = 0,$$

follows that the axes Oy and Oz must be main axes of inertia of the cross section;

Starting from 5.4:

$$\int_{(A)} y^2 \cdot dA = I_z,$$

the moment of inertia axial to the neutral axis Oz of the entire cross section.

The axes (Oy and Oz) of the section which pass through the center of gravity and Oy which is the axis of symmetry are main central axes of inertia. If the neutral surface is intersected with a normal plane, **the bending axis** (Oz axis) of the section is obtained.

Taking the above into consideration, the rotation of the section is defined by the following relation:

$$\omega = \frac{1}{r} = \frac{M}{E \cdot I_z} . \qquad \textbf{(8.5)}$$

Therefore, the section rotation is equal to **curvature ($\frac{1}{r}$)** and is directly proportional to the bending moment and inversely proportional to the **bending rigidity ($E \cdot I_z$).**

If in relation (8.5) the relation (8.2) is taken into account, the expression of the normal stress becomes:

$$\sigma = \frac{M}{I_z} \cdot y . \qquad \textbf{(8.6)}$$

This is L. M. H. Navier’s formula and shows that **the value of the normal bending stress is a linear function related to the ordinate of the point, relative to the neutral axis**. Navier's relation expresses a linear distribution of the stresses: zero in the neutral axis and maximum and minimum values in the boundary fibers (fig. 8.2, c). The maximum stress in the section is:

$$\sigma_{\max} = \frac{M}{I_z} \cdot y_{\max} = \frac{M}{w_z} . \qquad \textbf{(8.7)}$$

The geometric size was introduced in formula (8.7) (see 5.7):

$$w_z = \frac{I_z}{y_{\max}} , \qquad \textbf{(8.8)}$$

which is called **modulus of axial strength**.

Although Navier's equation has been solved and corresponds to the pure bending load, it is also used to calculate the normal stresses at the beams subjected to simple bending.

If the bending axis is not the axis of symmetry, then both the maximum elongation stress and the maximum compressive stress are determined.

$$\sigma_1 = \frac{M}{W_{z1}} \quad \text{and} \quad \sigma_2 = \frac{-M}{W_{z2}} \qquad \textbf{(8.9,a)}$$

$$W_{z1} = \frac{I_z}{y_1} \quad \text{and} \quad W_{z2} = \frac{I_z}{y_2} \qquad \textbf{(8.9,b)}$$

In the above relations w_{z1} and w_{z2} are the strength moduli.

8.3. Calculation of Bending Strength

The relations deduced above are used to solve the problems of the strength of materials: verification, calculation of maximum applicable load and dimensioning. Solving these problems is done by meeting the strength condition $\boldsymbol{\sigma_{max} \leq \sigma_a}$. The relations for calculating the bending strength are deduced from relation (8.8) and are the following:

- for verification:

$$\sigma_{\max} = \frac{M_{i\max}}{w_z} \leq \sigma_a \; , \qquad \textbf{(8.10)}$$

- for calculating the maximum applicable load:

$$M_{icap} = w_{zef} \cdot \sigma_a \; , \qquad \textbf{(8.11)}$$

- for dimensioning:

$$w_{znec} = \frac{M_{i\max}}{\sigma_a} \; . \qquad \textbf{(8.12)}$$

Relations (8.10), (8.11) and (8.12) apply to the most loaded section (the hazardous section). In the case of beams (cantilevers) of constant section, this corresponds to the section **where the bending moment is maximum at absolute value**. At the beams (cantilevers) with gradual section variation, a hazardous section is determined based on the bending moment diagram, for each segment, for which the strength calculation is then performed.

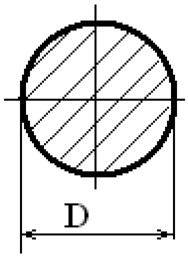
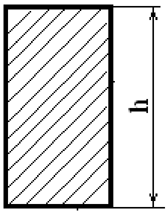
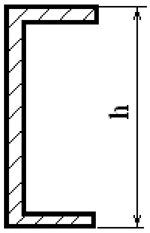
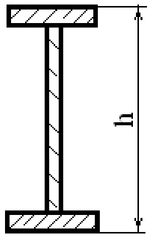
8.4. Rational Shapes of Cross Sections for Bending – S19

The higher the axial strength modulus $\mathbf{w_z}$, the better a beam (cantilever) withstands the bending load. The value of the axial strength modulus depends not only on the size of the section, but also on its shape. **The shape of the section is all**

the more rational as the strength modulus has a higher value for a lower material consumption.

In other words, a section is all the more rational as the ratio between the axial strength modulus and the section area is higher. Table 8.1 contains the values of this ratio for some habitual shapes of sections.

Table 8.1

Shape of the section				
W_z / A	0,125 D	0,167 h	≈ 0,26 h	≈ 0,3 h

This table shows that the sections of laminated profiles I and U, widely used in metallic constructions, are much more rational than the circular and rectangular sections. In the case of these profiles, the section is rationally used as most of the material is concentrated where the stresses are high.

These profiles must be **acted upon by bending moments in the direction of the main axis Oz**.

The circular section has the advantage of withstanding equally well with respect to any central axis and is therefore **used especially for building machine shafts**. In this case the forces maintain their position in space, whereas the shaft rotates, which must withstand as well in any position.

In the case of materials that withstand compression better than elongation (e.g. cast iron), those sections that are not symmetrical with respect to the bending axis are more rational (e.g. section T, the trapezoidal

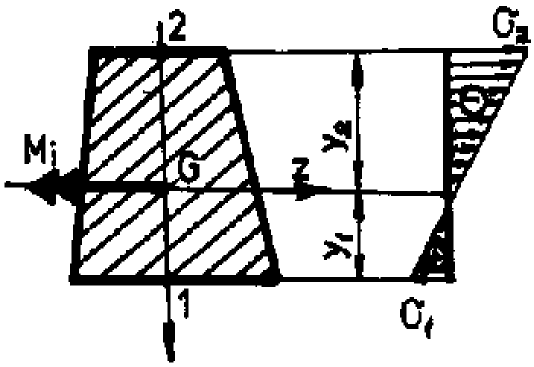


Fig. 8.3

section fig. 8.3). The beam made of brittle materials must be placed in such a manner that the compressive stresses must be higher than the tensile stresses. In this case, both the tensile and compressive strength conditions must be met.

$$\sigma_1 = \frac{M_i}{I_z} \cdot y_1 \le \sigma_{at}; \quad \sigma_2 = \frac{M_i}{I_Z} \cdot y_2 \le \sigma_{ac}. \qquad \textbf{(8.14)}$$

By drawing the ratio of these two relations the optimal dimensions of the section are obtained:

$$\frac{y_1}{y_2} = \frac{\sigma_{at}}{\sigma_{ac}}. \qquad \textbf{(8.15)}$$

Exercise 8.1 Consider the beam in figure 9.4, which can be made in 3 constructive variants, all of the same weight, and determine the maximum applicable load for each variant, when the allowable stress is σ_a = 150 MPa and a = 40 mm.

The areas of the sections are equal in the three situations, and the axial strength

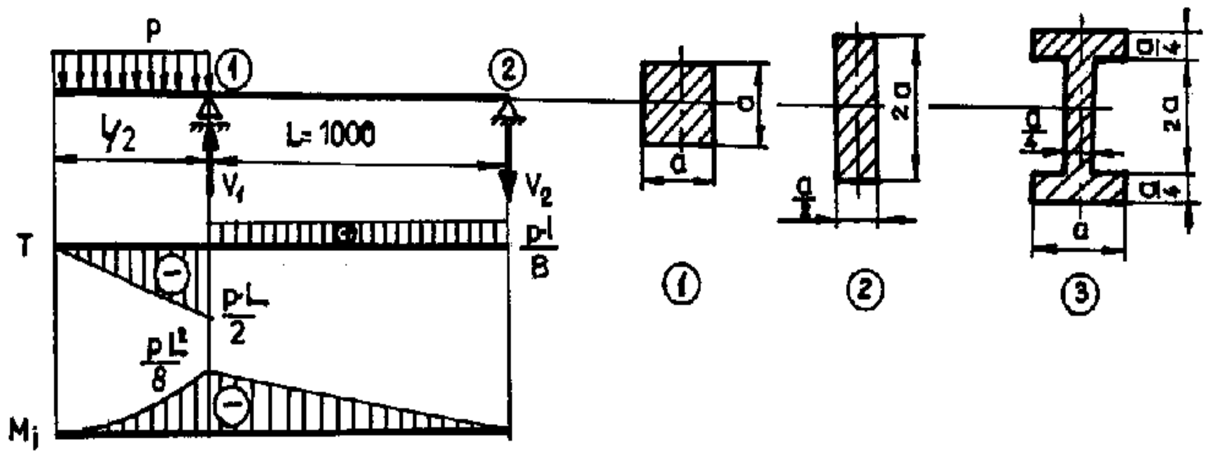


Fig. 8.4

moduli have the following values:

$$W_{z1} = \frac{a^3}{6}, \qquad W_{z1} = \frac{a^3}{3},$$

$$W_{z3} = \frac{4}{5a} \cdot \left[\frac{a}{12} \cdot \left(\frac{5a}{2} \right)^3 - \frac{3a}{4} \cdot \frac{(2a)^3}{12} \right] = \frac{77}{120} \cdot a^3.$$

From the strength condition:

$$M_{i\,max} = \frac{p \cdot L^2}{8} = W_z \cdot \sigma_a,$$

results the value of the force for the three constructive variants:

$$p_{1cap} = \frac{8 \cdot a^3}{6L^2} \cdot \sigma_a = \frac{8 \cdot 40^3 \cdot 150}{6 \cdot 1000^2} = 12.8N/mm = 12.8kN/m,$$

$$p_{2cap} = \frac{8a^3}{3L^2} \cdot \sigma_a = \frac{8 \cdot 40^3 \cdot 150}{3 \cdot 1000^2} = 25.6N/mm = 25.6kN/m,$$

$$p_{3cap} = \frac{8 \cdot 77 \cdot a^3}{120 \cdot L^2} \cdot \sigma_a = \frac{8 \cdot 77 \cdot 40^3 \cdot 150}{120 \cdot 1000^2} = 49.28N/mm = 49.28kN/m.$$

The section corresponding to the third variant best withstands the bending stress, the variant being 3.85 times more resistant than the first variant. Therefore, **by judiciously choosing the shape of the section, significant material reductions are made.**

Exercise 8.2. Dimension a cast iron beam of σ_{at} = 30 MPa and σ_{ac} = 90 MPa, length l = 1300 mm and having a T-shaped section, with $t = \frac{b}{9}$, loaded by a force P = 24 kN, (fig.8.5).

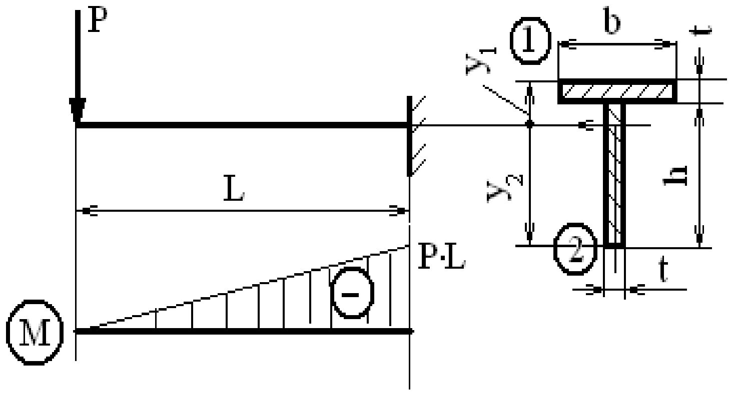


Fig. 8.5

Solution: In points 1 and 2 of the section the maximum stress must not be greater than the allowable elongation stress and the compressive one respectively.

$$\sigma_1 = \frac{M_i}{I_z} \cdot y_1 \le \sigma_{at}, \qquad \sigma_2 = \frac{M_i}{I_z} \cdot y_2 \le \sigma_{ac}.$$

The ordinates y_1 and y_2 measured from the neutral axis (the axis passing through the center of gravity) result from the expressions:

$$y_1 = \frac{b \cdot t \cdot \frac{t}{2} \cdot + h \cdot t \cdot \left(t + \frac{h}{2}\right)}{(b+h) \cdot t} = \frac{b \cdot t + 2h \cdot t + h^2}{a \cdot (b+h)} = \frac{b^2 + 2b \cdot h + 9h^2}{18 \cdot (b+h)},$$

$$y_2 = \frac{h \cdot t \cdot \frac{h}{2} \cdot + b \cdot t \cdot \left(h + \frac{t}{2}\right)}{(b+h) \cdot t} = \frac{h^2 + 2b \cdot h + bt}{a \cdot (b+h)} = \frac{b^2 + 18 \cdot b \cdot h + 9 \cdot h^2}{18 \cdot (b+h)}.$$

Relation 8.15 leads to:

$$\frac{\sigma_1}{\sigma_2} = \frac{\sigma_{at}}{\sigma_{ac}} = \frac{y_1}{y_2} \text{ or } \frac{b^2 + 2b \cdot h + 9h^2}{b^2 + 18b \cdot h + 9h^2} = \frac{1}{3}.$$

From this relation, the following results:

b^2 - 6 bh + $9h^2$ = 0, with the solution compatible with problem: b = 3h.

With this solution, the dimensions of the section, expressed according to the thickness **t**, are the following:

b = 9t; h = 3t; y_1 = t; y_2 = 3t.

The moment of inertia of the section is:

$$I_z = \frac{t \cdot (3t)^3}{12} + t \cdot (3t) \cdot \left(\frac{3t}{2}\right)^2 + \frac{(9t) \cdot t^3}{12} + t \cdot (9t) \cdot \left(\frac{t}{2}\right)^2 = 12t^4$$

and the axial strength moduli are:

$$W_{z1} = \frac{I_z}{y_1} = \frac{12 \cdot t^4}{t} = 12 \cdot t^3 \qquad W_{z2} = \frac{I_z}{y_2} = \frac{12 \cdot t^4}{3t} = 4 \cdot t^3$$

From the condition of bending strength $M_{imax} = W_z \cdot \sigma_a$, the following thickness is obtained:

$$t_{nec} = \sqrt[3]{\frac{M_{max}}{12 \cdot \sigma_{at}}} = \sqrt[3]{\frac{12 \cdot 10^3 \cdot 1300}{12 \cdot 30}} = 44.25mm$$

The following values are chosen: t = 45 mm; b = 405 mm; h = 135 mm.

8.5. Tangential Stresses in the Cross Sections (Cantilevers) Subjected to Simple Bending – S20

The cross section of a beam (cantilever), subjected to simple bending, is acted upon by the following stresses: bending moment and shear force. The simply supported beam, loaded by the traverse force P, (fig. 8.6, a), is subjected to simple bending. An element of this beam is selected having length **dx** (fig.8.6, b). Stresses **T, M** and respectively **T** and **M + dM** occur in the cross-sections**.**

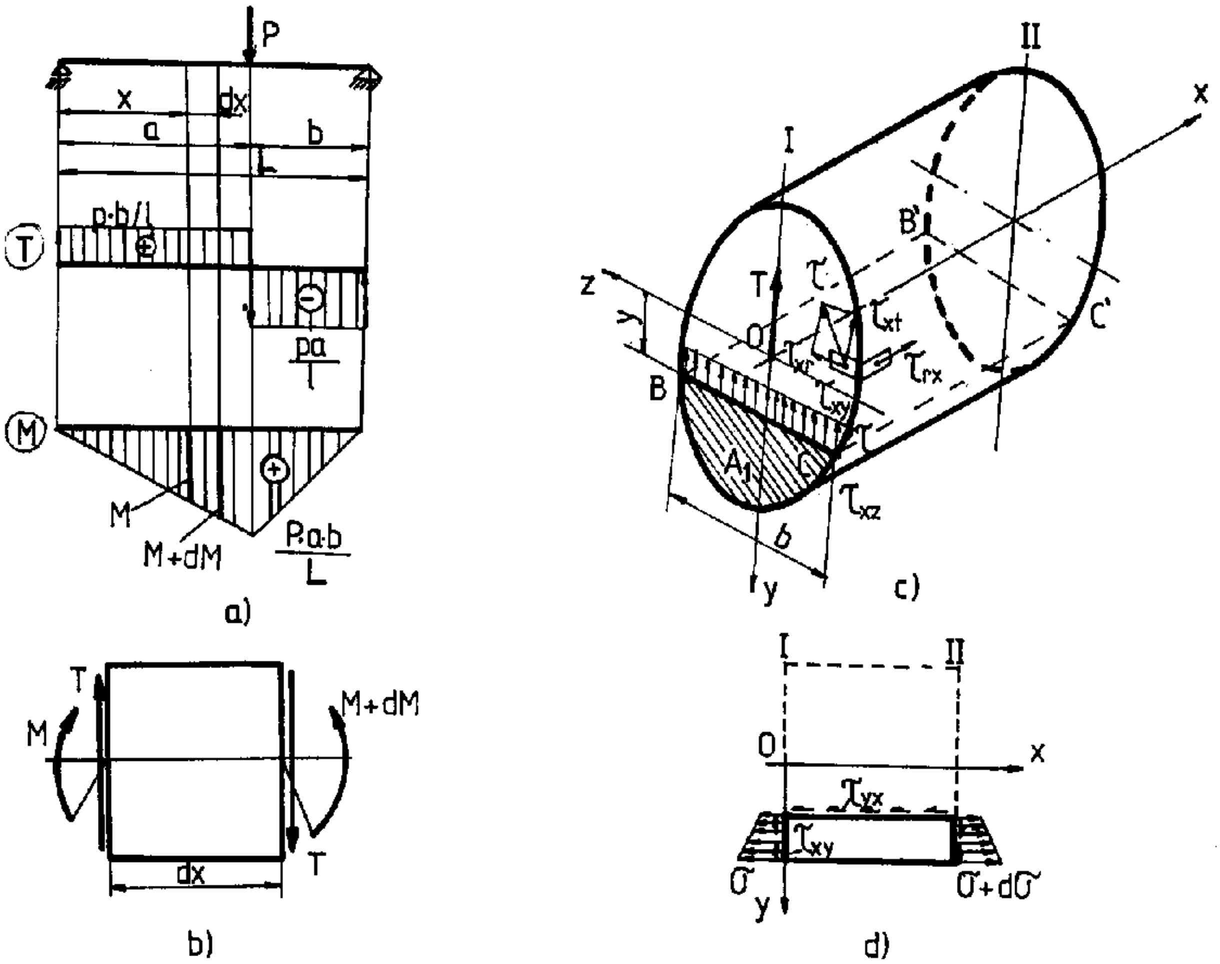


Fig. 8.6

It is admitted that the section of the beam is symmetrical to the axis Oy (fig. 8.6 c) and constant throughout its length L. The beam is made of a homogeneous and isotropic material that obeys Hooke's law. The shearing force is directed along the Oy axis.

The bending moments **M** and **M + dM** will produce normal stresses **σ** and respectively **σ + dσ** in the two sections, their distribution on the section being given by Navier's equation:

$$\sigma = \frac{M_i}{I_z} \cdot y, \text{ respectively } \sigma + d\sigma = \frac{M_i + dM_i}{I_z} \cdot y, \qquad \textbf{(8.16)}$$

and is shown in figure (8.6,d).

The shearing force **T** causes tangential stresses. Their distribution in the cross section is yet unknown. **The tangential tension, in the vicinity of the points on the contour must be tangent to the contour.** If at one point on the contour the tangential stress **τ** had a random direction (fig. 8.6c), then it would be separated into two components: one $\boldsymbol{\tau_{xt}}$ tangent to the contour and another one $\boldsymbol{\tau_{xr}}$ normal to the contour. Component $\boldsymbol{\tau_{xr}}$ should correspond, according to the principle of duality of the tangential stresses to a stress $\boldsymbol{\tau_{rx}}$ located on the outer surface of the beam and oriented along the bar. As the beam is subjected to simple bending and longitudinal frictional forces do not act upon the beam, it turns out that the two components $\boldsymbol{\tau_{rx}}$ and $\boldsymbol{\tau_{xr}}$ (on the outer surface and in the cross section) are null. Therefore, the tangential stress **τ** is equal to component $\boldsymbol{\tau_{xt}}$ **($\tau = \tau_{nt}$)**, which means that on the points in the vicinity of the contour there are only tangential stresses tangent to the contour.

We consider a line BC parallel to the bending axis Oz (located on its y-coordinate). We mark the area of the cross section below line BC cu $\mathbf{A_1}$. The length of segment BC is marked with b. In points B and C the tangential stresses **τ** are tangent to the contour and can be decomposed into a component $\boldsymbol{\tau_{xy}}$ perpendicular to the bending axis Oz and a component $\boldsymbol{\tau_{xz}}$ parallel to the bending axis. According to **the D.I. Juravski's hypothesis, it is admitted that the values of component $\boldsymbol{\tau_{xy}}$ are equal in the vicinity of all points on the BC line**.

We consider a plane parallel to the axis of the beam, which contains the segment **BC = b**. This plane (BCC'B') intersects the element **dx** after a rectangular surface with dimensions **b** and **dx**. Both the tangential stresses $\boldsymbol{\tau_{xy}}$ caused by the shearing force T, as well as the normal stresses **σ and σ+dσ** caused by the bending moment **M** on the left and **M+dM** on the right, act upon the bottom side of the considered plane (below ordinate y).

The equation of projections of the stresses on the element under the BCCăBă plane on axis Ox is the following:

$$\int_{A_1}(\sigma + d\sigma)\cdot dA - \int_{A_1}\sigma \cdot dA - \tau_{xy}\cdot b\cdot dx = 0$$

Taking into account relations (8.16), the equation becomes:

$$\int_{A_1}\frac{M_i + dM_i}{I_z}\cdot y\cdot dA - \int_{A_1}\frac{M_i}{I_z}\cdot y\cdot dA + \tau_{xy}\cdot b\cdot dx = 0\,,$$

The value of the tangential stress is:

$$\tau_{xy} = \frac{1}{b \cdot I_z} \cdot \frac{dM_i}{dx} \cdot \int_{A1} y \cdot dA .$$

Considering that $\frac{dM}{dx} = T$ is the shearing force in the section and $\int_{A1} y \cdot dA = S_z$ **is the static moment of surface A_1, (below line BC) related to axis Oz,** we obtain:

$$\tau = \tau_{xy} = \tau_{yx} = \frac{T \cdot S_z}{b \cdot I_z} , \qquad \textbf{(8.17)}$$

This relation is known as **Juravski's formula.**

According to Juravski's formula the value of the tangential stress in a given cross-section depends on the value of ratio $\mathbf{S_z/b}$, which means that $\boldsymbol{\tau_{xy}}$ is a function of ordinate **y**. On the lower and upper edge of the section these stresses are null because $\mathbf{A_1 = 0}$.

8.6. Variation of the Tangential Stresses in Different Cross Sections

a) The rectangular cross section (fig 8.7).

In this case, the width **b** is constant over the section height. The quantities of Juravski's formula have the following values:

$$I_z = \frac{b \cdot h^3}{12}; \qquad A_1 = (\frac{h}{2} - y) \cdot b; \qquad e = \frac{1}{2} \cdot (\frac{h}{2} + y);$$

$$S_z = A_1 \cdot e = \frac{b}{2} \cdot (\frac{h^2}{4} - y^2) = \frac{b \cdot h^2}{8} \cdot (1 - \frac{4y^2}{h^2}) . \qquad \textbf{(8.18)}$$

By substituting these quantities in relation (8.17), we obtain:

$$\tau = \frac{T \cdot S}{b \cdot I_z} = \frac{T \cdot \frac{b \cdot h^2}{8} \cdot (1 - 4\frac{y^2}{h^2})}{b \cdot \frac{b \cdot h^3}{12}} = \frac{3}{2} \cdot \frac{T}{b \cdot h} \cdot (1 - 4\frac{y^2}{h^2}) = \frac{3}{2} \cdot \frac{T}{A} \cdot (1 - 4\frac{y^2}{h^2}) \qquad \textbf{(8.19)}$$

where the area of the cross section was noted with $\mathbf{A = b \cdot h}$.

Relation (9.19) shows that **the tangential stresses vary parabolically** over the height of the section. The maximum tangential stress is reached in the vicinity of the neutral axis, for y = 0 and has the value:

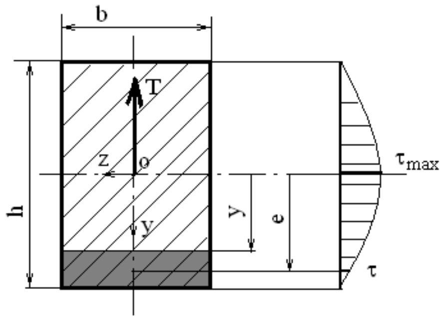


Fig. 8.7

$$\tau_{max} = \frac{3T}{2A}. \qquad \textbf{(8.20)}$$

Thus, the maximum value of the tangential stress, in the case of shearing the rectangular section beams, is 50% higher than the value obtained by conventional shear calculation.

b) The circular cross section.

We consider a circular section of diameter **d** (fig. 8.8). In order to calculate the static moment, we consider an area element **dA**, of width **b** and height **dy**, located on the ordinate y.

Width BC of section A_1 is:

$b = 2\frac{d}{2} \cdot \sin\alpha = d \cdot \sin\alpha$, and the ordinate is

$y = \frac{d}{2} \cdot \cos\alpha$, so that

$dy = -\frac{d}{2} \cdot \sin\alpha \cdot d\alpha$.

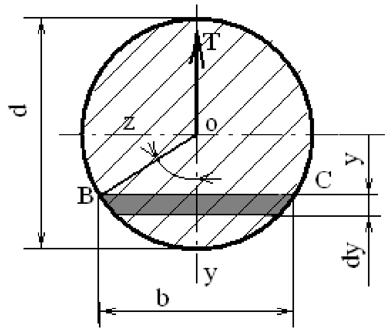


Fig. 8.8

It results that the elementary area is:

$$dA = b \cdot dy = -\frac{d^2}{2} \cdot \sin^2\alpha \cdot d\alpha.$$

The static moment of section $\mathbf{A_1}$, below ordinate **y** will be:

$$S_z = \int_{A_1} y \cdot dA = \int_{-\alpha}^{\alpha} \frac{d}{2} \cdot \cos\alpha \cdot \left(-\frac{d^2}{2} \cdot \sin^2\alpha\right) \cdot d\alpha = \frac{d^3}{12} \cdot \sin^3\alpha.$$

Taking into account that:

$$A = \frac{\pi \cdot d^2}{4}; \qquad I_z = \frac{\pi \cdot d^4}{64}; \qquad \sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{4 \cdot y^2}{d^2},$$

the value of the tangential stress will be:

$$\tau = \frac{T \cdot \frac{d^3}{3} \cdot sin^3 \alpha}{d \cdot sin\alpha \cdot \frac{d^4}{64}} = \frac{16\, sin^2 \alpha}{3d^2} = \frac{4}{3} \cdot \frac{T}{A} \cdot (1 - \frac{4y^2}{d^2}). \qquad \textbf{(8.21)}$$

The value of the maximum tangential stress is obtained in a similar way as for the rectangular section for **y = 0** and reaches the value:

$$\tau = \frac{4}{3} \cdot \frac{T}{A}. \qquad \textbf{(8.22)}$$

Relation (8.21) proves that the tangential stresses also vary parabolically as in the case of the rectangular section.

Exercise 8.6. Draw the stress variation diagrams in the hazardous section for the beam in figure 8.9.

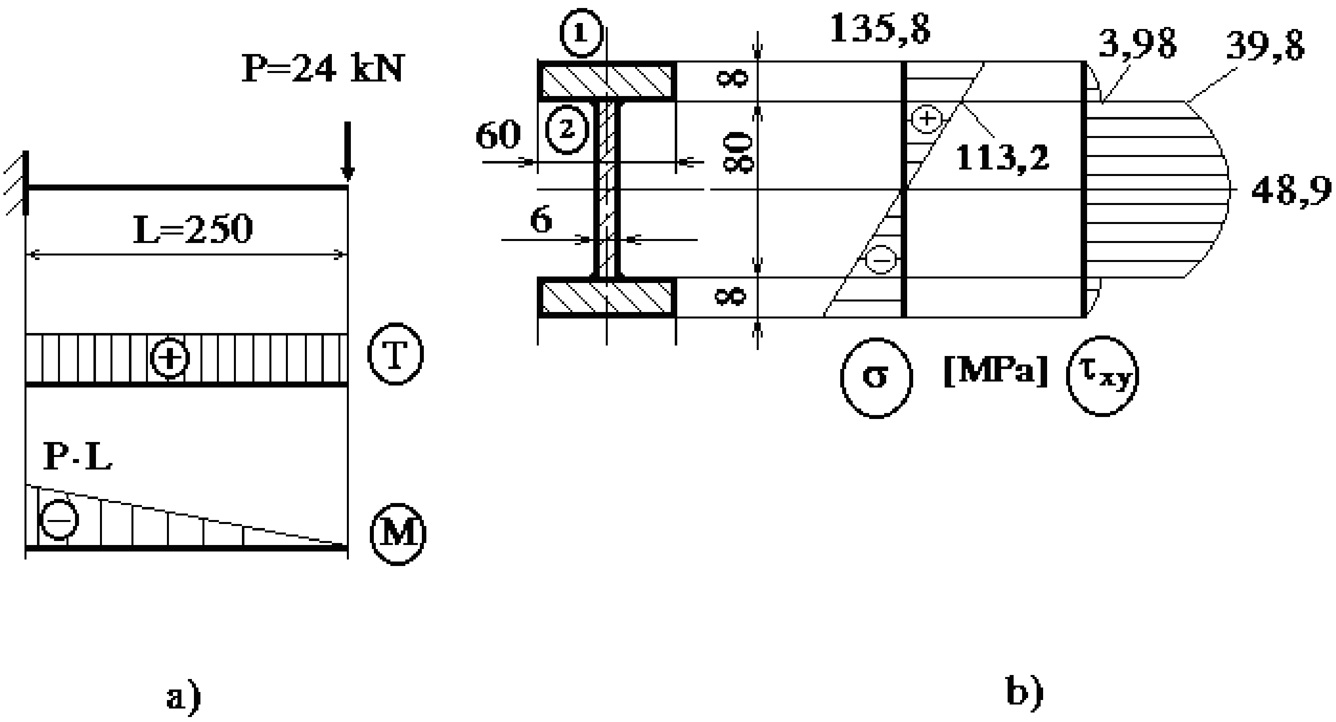


Fig. 8.9

The geometric properties of the section are:

$$I_z = \frac{6 \cdot 9.6^3}{12} - \frac{5.4 \cdot 8^3}{12} = 212 cm^4, \qquad W_z = \frac{I_z}{|y_{\max}} = \frac{212}{4,8} = 44.17 cm^3,$$

$$S_{z1} = 0, \qquad S_{z2} = S_{z3} = 6 \cdot 0.8 \cdot 4.4 = 21.12 cm^3,$$

$$S_{zG} = S_{z3} + 4 \cdot 0.6 \cdot 2 = 25.92 cm^3,$$

The stresses corresponding to the fixed end section are the following:

$$\sigma_{\max} = \frac{M_{i,\max}}{W_z} = \frac{24 \cdot 10^3 \cdot 250}{44.17 \cdot 10^3} = 135.8 MPa \ ,$$

$$\sigma_2 = \frac{M_{i,\max}}{I_z} \cdot y^2 = \frac{24 \cdot 10^3 \cdot 250}{212 \cdot 10^4} \cdot 40 = 113.2 MPa \, ,$$

$$\tau_{xy1} = 0 \, ; \qquad \tau_{2xy} = \frac{T \cdot S_{z2}}{b_2 \cdot I_z} = \frac{24 \cdot 10^3 \cdot 21.12 \cdot 10^3}{60 \cdot 212 \cdot 10^4} = 3.985 MPa;$$

$$\tau_{3xy} = \frac{T \cdot S_{z3}}{b_3 \cdot I_z} = \frac{24 \cdot 10^3 \cdot 21.12 \cdot 10^3}{6 \cdot 212 \cdot 10^4} = 39.85 MPa \, ,$$

$$\tau_{Gxy} = \frac{T \cdot S_{zG}}{b \cdot I_z} = \frac{24 \cdot 10^3 \cdot 25.92 \cdot 10^3}{6 \cdot 212 \cdot 10^4} = 48.91 MPa \, ,$$

and their variation is shown in figure (8.9,b).

9. COMPOUND LOADS – S21

9.1. Introductory Notions – S21

The simple loads acting on the mechanical member (MM) have been studied up to this point. In engineering practice, there are frequent cases when two or more simple loads are present simultaneously. **The simultaneous presence of two or more stresses in the cross section of a mechanical member produces a compound load**.

In the case of compound loads, each force will cause a stress in the section, respectively a strain, which can be calculated with the formulas learned in the chapter related to simple loads. However, the problem arises of cumulating these stresses or strains and establishing the limit state for these cases.

9.2. Limit State – S21

Often the limit of proportionality or sometimes that of elasticity and, in some cases, the flow limit, are exceeded, producing permanent deformations (non-elastic therefore irreversible). This is the situation when one says about the MM that **it does not endure**. **The fact that it fails does not imply that the MM fractures, but that it exceeds a limit state**.

An MM is said to have reached the limit state when it no longer meets the technical conditions imposed by normal operation, i.e. its operation becomes impossible.

The limit states can be classified into two groups:

I - **limit states of total depletion of the load capacity**, which can be characterized by:

a) the fracture of the MM;

b) reaching the flow limit on the entire section of the MM and

c) the occurrence of the elastic instability phenomenon (buckling).

II - **limit states of functional failure**, characterized by:

a) the occurrence of elastic or non-elastic deformations which exceed the allowable values;

b) the occurrence of inadmissible vibrations.

The proper functioning of the MM is compromised by the existence of any of the above limit states.

9.3. Equivalent Stress – S22

The verdict given by the engineers that **an MM does not endure,** means that a certain limit state has been exceeded. From here on we will use the notion of limit state to refer to the point where a mechanical or elastic property of the material is reached when the basic hypothesis of strength of materials are considered achieved, namely the relations of the theory of elasticity are applicable. This limits the notion of limit state to the linear - elastic domain.

Five criteria are considered when determining the limit state:

I. the maximum normal stress;

II. the maximum specific elongation;

III. the maximum tangential stress;

IV. the total specific energy of maximum deformation;

V. the specific energy of changing the maximum shape.

These five criteria have proven worthy for two reasons:

a) The tensile-compressive tests can determine the values of the mechanical properties corresponding to the limit state that must not be exceeded;

b) Between the boundary stress determined by the tensile-compressive test (which must not be exceeded) and the five criteria which determine the limit state, simple relations can be established.

If we consider the limit of proportionality as a limit state, the other limit state criteria can be written according to σ_p:

$$\sigma_{max} \le \sigma_p; \quad \varepsilon_p = \frac{\sigma_p}{E}; \quad \tau_p = \frac{\sigma_p}{2}; \quad U_{1p} = \frac{\sigma_p^2}{2E}; \quad U_{1fp} = \frac{1+\nu}{3E} \cdot \sigma_p^2. \qquad (9.1)$$

The spatial state of stress in a point in the MM can, by hypothesis, be equated with a uniaxial state of stress. The equivalence is done by using one criterion of the above mentioned five. This can be summarized by the figure below.

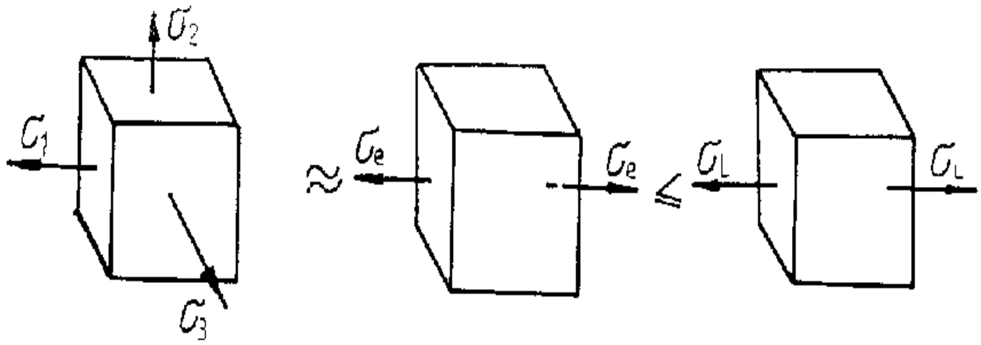


Fig. 9.1

If the limit state for tension or compression is known, the five traditional theories of strength can be asserted, establishing the conditions under which the limit state is reached in a point of a spatially loaded mechanical member. The verification of the limit state is done by determining a conventional stress, for a critical state of stress in a point, based on the accepted strength hypothesis, called **equivalent stress**, which must meet the relation:

$$\sigma_{ech} \le \sigma_L. \qquad (9.2)$$

This inequality can also be written as an equality, under limit conditions:

$$\sigma_{ech} = \frac{\sigma_L}{c}, \qquad (9.3)$$

where, c – is the corresponding safety coefficient.

9.4. The Traditional Theories of Strength – S23

Depending on the five parameters chosen for reaching the limit state, there are five theories (hypotheses) of strength.

9.4.1 The theory of the maximum normal stress

It was initially formulated by Galileo Galilei and reformulated by Rankine.

The limit state is reached in one point of an MM when the maximum normal stress in that point becomes equal to the boundary normal stress of the simple tensile or compressive state of the material the respective MM is made of.

This theory can also be expressed through the following relations:

$$\begin{aligned} -\sigma_{Lc} \le \sigma_1 \le \sigma_{Lt}, \\ -\sigma_{Lc} \le \sigma_2 \le \sigma_{Lt}, \\ -\sigma_{Lc} \le \sigma_3 \le \sigma_{Lt}, \end{aligned} \qquad (9.5)$$

which can be represented by a cube for the spatial state (fig.9.2, a) or a square for the plane state of stress (fig.9.2,b).

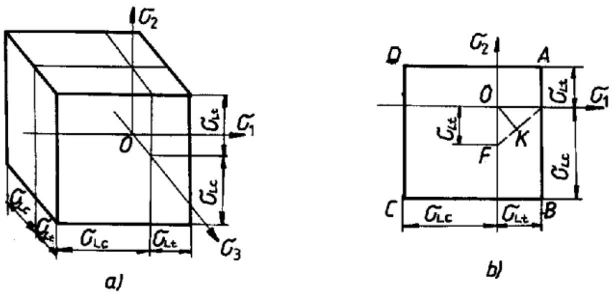


Fig. 9.2

If $\sigma_{Lt} \ne \sigma_{Lc}$, the origin of the axis system is not in the centroid of the cube (or of the square). This theory does not fully correspond to reality because in the three-dimensional state of compression ($\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_L$), in which the body cannot be destroyed, it must result $\sigma_{Lc} = \infty$.

Also, in case of shear, for which the limit stress is $\tau_L = \sigma_L / 2$, which corresponds to point K, inside the square and not to point B, which is the limit according to this theory.

Due to these inconsistencies, the theory of the maximum normal stress can be used with caution only for states of stress where the fracture occurs by tearing (it is a theory of tearing).

For the most unfavorable state of stress in a point on the MM, the equivalent stress, according to the theory of the maximum normal stress, is:

$$\boldsymbol{\sigma}_{\mathbf{ech}} = \mathbf{max}\left\{ |\boldsymbol{\sigma}_1|;\ |\boldsymbol{\sigma}_2|;\ |\boldsymbol{\sigma}_3| \right\} \le \boldsymbol{\sigma}_{\mathbf{L}}\,. \qquad \mathbf{(9.6)}$$

9.4.2. The theory of the maximum specific elongation

This theory was issued by Barré de Saint-Venant. According to this theory it is considered that the destruction of the mechanical member is caused by the maximum specific elongations. **In a point of a MM the limit state is reached when the maximum specific elongation, from that point, becomes equal to the value of the specific limit elongation for simple elongation or compression.**

$\varepsilon_{max} \le \varepsilon_L = \dfrac{\sigma_L}{E}$, or expressed in terms of stresses:

$$-\sigma_{LC} \le \sigma_1 - \nu\cdot(\sigma_2+\sigma_3) \le \sigma_{Lt}\,,$$
$$-\sigma_{LC} \le \sigma_2 - \nu\cdot(\sigma_3+\sigma_1) \le \sigma_{Lt}\,, \qquad \mathbf{(9.7)}$$
$$-\sigma_{LC} \le \sigma_3 - \nu\cdot(\sigma_1+\sigma_2) \le \sigma_{Lt}\,.$$

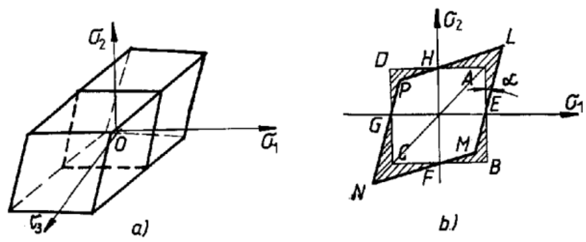


Fig. 9.3

Relations (9.7) express the boundary surface which is in this case, a spatially inclined parallelepiped (fig.9.3, a). For the plane state of stress the rhomb in figure (9.3, b) is obtained, which results from sectioning the parallelepiped with plane σ_3□= 0.

Angle α at which the sides of the rhomb in the second theory are inclined, as compared to the square representing the first theory is given by relation: **α□=arctg(ν)** This theory has almost the same weaknesses as the first one. That is why it can be applied, with good results to brittle materials, as a hypothesis of tearing.

The equivalent stress, in this case for the spatial state, is expressed by the relation:

$$\sigma_{ech} = max\left\{\begin{matrix} |\sigma_1 - \nu\cdot(\sigma_2+\sigma_3)| \\ |\sigma_3 - \nu\cdot(\sigma_1+\sigma_2)| \end{matrix}\right\} \le \sigma_L \qquad \textbf{(9.8)}$$

9.4.3. The theory of the maximum tangential stress

This theory was formulated by Coulomb and according to it the limit state appears through sliding in the plane acted upon by the maximum tangential stress. Tresca rephrased the theory, stating that **the limit state in one point of a MM is reached when the maximum tangential stress becomes equal to the value of the tangential stress (□L) from the simple tensile or compressive load**.

This theory can be expressed by: $\tau_{max} \le \frac{\sigma_L}{2}$, a condition met by:

$$-\tau_L \le \tau_1 \le \tau_L\,; \qquad -\tau_L \le \tau_2 \le \tau_L\,; \qquad -\tau_3 \le \tau_3 \le \tau_L\,.$$

considering that $\tau_1 = \pm\frac{\sigma_2-\sigma_3}{2}$; $\tau_2 = \pm\frac{\sigma_1-\sigma_3}{2}$ and $\tau_3 = \pm\frac{\sigma_1-\sigma_2}{2}$ we obtain:

$$-\sigma_L \le \sigma_1-\sigma_2 \le \sigma_L\,; \qquad -\sigma_L \le \sigma_1-\sigma_3 \le \sigma_L\,; \; -\sigma_L \le \sigma_2-\sigma_3 \le \sigma_L\,. \qquad \textbf{(9.9)}$$

Relations (9.9) create, due to the equal sign between stresses, a regular hexagonal prism open at the ends. The axis of the prism is the trisector $\sigma_1 = \sigma_2 = \sigma_3$ The surface is open at both ends because both at triaxial compression $\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_L$ as well as at triaxial elongation $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_L$, the tangential stresses are null (fig.9.4.a) and there are no sidings.

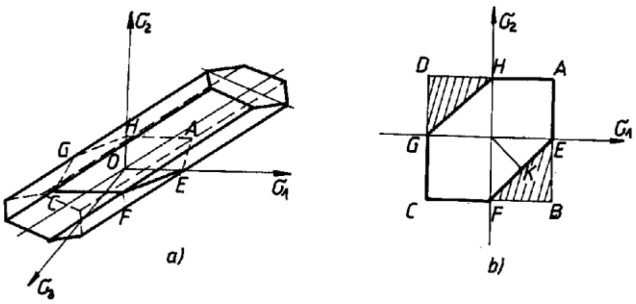


Fig. 9.4

According to this hypothesis, in these cases, the limit state is not reached and the MM is not destroyed. The conclusion is true only for uniform triaxial compression, but does not correspond to the reality of uniform triaxial elongation.

The plane state, which is a section with plane $\sigma_3 = 0$ (fig. 9.4), is represented by an irregular hexagon AEFCGH (fig. 9.4, b) and corresponds to theory **I** for $\sigma_1 \cdot \sigma_2 > 0$ and differs from it for $\sigma_1 \cdot \sigma_2 < 0$. In the case of pure shear, when $\sigma_1 = -\sigma_2 = \tau_{max}$, it is represented by point K by coordinates $\frac{\sigma_L}{2}$ and $-\frac{\sigma_L}{2}$.

This theory has been experimentally verified and it has been proven that it corresponds to reality except for the state of stress close to the triaxial elongation, when due to the fact that the tangential stresses are low, no sliding occurs.

Theory **III** is not perfect either because:

a) it does not take into account the influence of the normal stress in the plane of sliding;

b) it does not take into account the different strength of the materials to elongation and compression;

c) it neglects the effect of the intermediate stress (only two main stresses are taken into account).

The strength condition for this theory is expressed by the relation:

$$\sigma_{ech} = max\{(\sigma_1 - \sigma_2);(\sigma_1 - \sigma_3);(\sigma_2 - \sigma_3)\} \le \sigma_L .$$

If we take into consideration $\sigma_1 > \sigma_2 > \sigma_3$, the strength condition becomes:

$$\sigma_{ech} = |\sigma_1 - \sigma_3| \le \sigma_L , \qquad \textbf{(9.10)}$$

and it is therefore independent of the value of the intermediate normal stress σ_2.

9.4.4. The theory of the total strain energy

This theory was formulated by Haigh and states that: **at one point of a MM the limit state is reached**

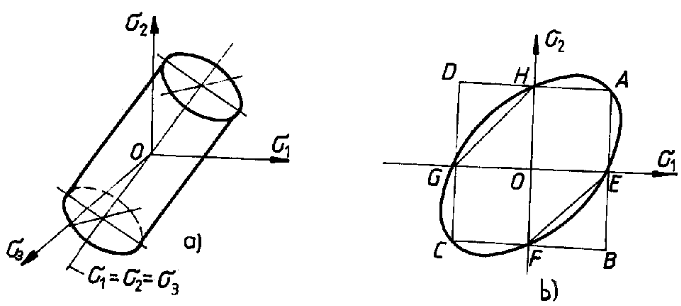


Fig. 9.5

when the specific strain energy becomes equal to the value of the specific strain energy corresponding to the simple elongation or compressive load, i.e.:

$\mathbf{U_1 \leq U_{1L}}$.

Expressing these strain energies according to stresses, the following inequality is obtained:

$$\frac{1}{2E}\cdot(\sigma_1^2+\sigma_2^2+\sigma_3^2-2\cdot\nu\cdot(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_3\cdot\sigma_1)\leq\frac{\sigma_L^2}{2E},$$

or simplifying by **(2E)** it results:

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\cdot\nu\cdot(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_3\cdot\sigma_1)\leq\sigma_L^2, \qquad \textbf{(9.11)}$$

relation which forms an ellipsoid.

For the plane state of stress, it is represented by an ellipse passing through the EFGH points (fig. 9.5). This strength theory is a tearing theory. It is used only for states of stress close to the triaxial elongation state: ($\frac{\sigma_1+\sigma_2+\sigma_3}{3}>0$).

The equivalent stress in this case is expressed with the relation:

$$\sigma_{ech}=\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-2\cdot\nu\cdot(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_3\cdot\sigma_1)}\leq\sigma_L. \qquad \textbf{(9.12)}$$

9.4.5. The theory of the shape variation specific energy

It was formulated by Huber - Hencky - Mises and takes into account only the shape variation specific energy.

According to this theory, at **one point of a MM the limit state is reached when the shape variation strain energy in that point becomes equal to the shape variation specific energy corresponding to the limit state for the simple elongation or compressive load.**

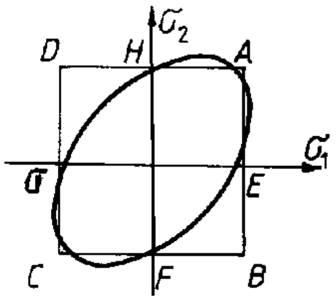


Fig. 9.6

$$U_{1f}\leq U_{1fL},$$

or, expressed according to stresses we obtain:

$$\frac{1+\nu}{6E}\cdot\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq\frac{1+\nu}{3E}\cdot\sigma_L^2,$$

and after simplifications, it becomes:

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\cdot\sigma_2-\sigma_2\cdot\sigma_3-\sigma_3\cdot\sigma_1\leq\sigma_L^2. \qquad \textbf{(9.13)}$$

Relation (9.15) represents a cylinder open at both ends, with the bisector $\sigma_1=\sigma_2=\sigma_3$ as axis (fig.9.6,a).

The normal section at the axis of the cylinder is a circle, and the cross section made with plane $\boldsymbol{\sigma_3=0}$, corresponding to the plane state of stress, is an ellipse circumscribed to the irregular hexagon of theory III, fig. 9.6b.

The equivalent stress in this case is expressed by the formula:

$$\sigma_{ech}=\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\cdot\sigma_2-\sigma_2\cdot\sigma_3-\sigma_3\cdot\sigma_1)}\leq\sigma_L. \qquad \textbf{(9.14)}$$

This theory is similar to reality except for the case of the uniform triaxial elongation state. It is superior to theory number III because it also takes into account the intermediate stress.

9.5. Particularities of the Theories of Strength

9.5.1. Plane state of stress

Replacing $\sigma_3 = 0$ in the above relations, it results the plane state of stress characterized only by the main stresses $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$.

The relations of the equivalent stresses become:

$$I)\ \sigma_{ech}=max\{|\sigma_1|;|\sigma_2|\}\leq\sigma_L;$$

$$II)\ \sigma_{ech}=max\{|\sigma_1-\nu\cdot\sigma_2|;|\sigma_2-\nu\cdot\sigma_1|\}\leq\sigma_L;$$

$$III)\ \sigma_{ech}=\sigma_1-\sigma_2\leq\sigma_L;$$

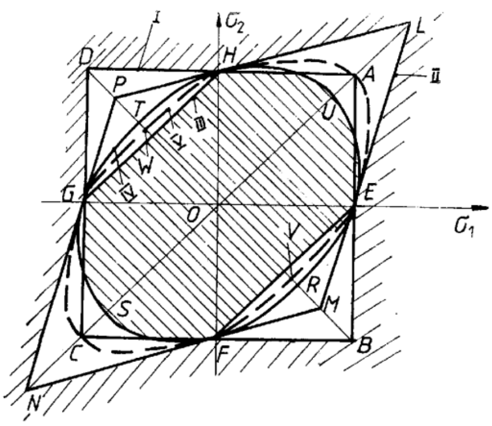


Fig. 9.7

$$IV)\,\sigma_{ech} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\nu \cdot \sigma_1 \cdot \sigma_2} \le \sigma_L\,;$$

$$V)\,\sigma_{ech} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \cdot \sigma_2} \le \sigma_L\,. \qquad \textbf{(9.15)}$$

These relations represented the following in figure 9.7:

- the square ABCD – according to theory **I**;
- the rhomb LMNP - according to theory **II**;
- the irregular hexagon AEFCGHA - according to theory **III**;
- the ellipse ERFSGTHUE - according to theory **IV**;
- the ellipse EVFCGWHAE - according to theory **V**.

It can be observed from this figure that in the points on the axes, that is, at simple elongation or compression, all the strength hypotheses coincide. The interior shaded surface represents the plane states $\boldsymbol{\sigma_1, \sigma_2}$ which do not exceed the limit state according to all three hypotheses, and the outer shaded surface represents the states of stress which, according to all hypotheses, lead to exceeding the limit state. The not-shaded surface represents the undefined area, not established by the different theories of strength.

9.5.2. Applying the theories of strength on beams– S23

In the particular case of the beams, in the sections where there can only be normal stresses $\boldsymbol{\sigma = \sigma_x}$ and tangential stresses $\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2}$, the principal stresses are obtained with the relation:

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4 \cdot \tau^2}\,,$$

which replaced in relations (9.17), for □□= **0,3** result in the formulas:

$$I)\,\sigma_{ech} = 0{,}5 \cdot (|\sigma| + \sqrt{\sigma^2 + 4 \cdot \tau^2}) \le \sigma_L :$$

$$II)\,\sigma_{ech} = (\frac{1-\nu}{2} \cdot |\sigma| + \frac{1+\nu}{2} \cdot \sqrt{\sigma^2 + 4 \cdot \tau^2} = 0{,}35 \cdot |\sigma| + 0{,}65 \cdot \sqrt{\sigma^2 + 4 \cdot \tau^2} \le \sigma_L\,;$$

$$III)\,\sigma_{ech} = \sqrt{\sigma^2 + 4 \cdot \tau^2} \le \sigma_L\,;$$

$$IV)\,\sigma_{ech} = \sqrt{\sigma^2 + 2(1+\nu) \cdot \tau^2} = \sqrt{\sigma^2 + 2{,}6 \cdot \tau^2} \le \sigma_L\,;$$

$$V)\ \sigma_{ech} = \sqrt{\sigma^2 + 3 \cdot \tau^2} \le \sigma_L . \qquad \textbf{(9.16)}$$

9.6. Calculation of Strength of Beams Subjected to Compound Loads – S24

A compound load refers to the existence of two or more forces simultaneously acting on the beam, cases that are frequently encountered in engineering practice. But each force causes a stress, some normal, some tangential. Due to this fact, the compound loads can be studied taking into account the stresses that cause them.

According to the type of stress produced, the forces that cause the compound load are grouped into the following three groups:

a) **N** and **M** (**M_y** and **M_z**) which cause normal stresses;

b) **T** (**T_y** and **T_z**) and **M_t** which cause tangential stresses;

c) **N** and **T** or **N** and **M_t**, **M** and **M_t**, **M** and **M_t**, **N**, **M**, **M_t**, which cause both normal and tangential stresses.

In cases **a** and **b** when the stresses have the same direction they are algebraically summed up, and when they have different directions, they are geometrically summed up.

In case **c**, the two types of stresses **σ** and **τ** are not algebraically nor geometrically summed up, but only by using one of the theories of strength (the corresponding one).

According to the form of the section, group **c** of compound load is subdivided, for analysis purposes into two subgroups:

- circular or annular cross section beams, and

- arbitrary cross section beams.

9.7. Calculation of Strength of Circular and Annular Cross Section Shafts Subjected to Bending and Torsion – S25

Among the MMs subjected to a compound load where both normal and tangential stresses occur, a very high frequency in the engineering practice is observed in the case of shaft, driving axles, screws, and so on.

The shafts are machine parts that transmit through gears, riggers or couplings, torque moments and are subjected to simple bending. The strength of the shafts is calculated by taking into account only the moments of bending and torsion, neglecting the effect of the shearing force. Due to these moments, the maximum normal and tangential stresses that occur in the hazardous cross sections are determined with the following relations:

$$\sigma_{max} = \frac{M_i}{W_z} \quad \text{and} \quad \tau_{max} = \frac{M_t}{W_p}.$$

Considering that in a circular or annular cross section, $W_z = 2W_p$, the maximum stresses, expressed only according to the axial strength modulus, are:

$$\sigma_{max} = \frac{M_i}{W_z} \quad \text{and} \quad \tau_{max} = \frac{M_t}{2W_z}.$$

Since, both at bending and torsion, these stresses reach maximum value in the farthest points from the neutral axis (Oz in fig. 9.11), the equivalent stress is calculated for these points. Using the theory of strength number III (III, 9.16), the following is obtained:

$$\sigma_{ech} = \sqrt{\sigma^2 + 4 \cdot \tau^2} = \sqrt{\frac{M^2}{W_z^2} + 4\frac{M_t^2}{(2W_z)^2}} = \sqrt{\frac{M^2 + M_t^2}{W_z^2}} = \frac{M_{ech}}{W_z}.$$

Where, according to theory III, we noted with:

$$M_{ech} = \sqrt{M_i^2 + M_t^2},$$

a quantity which is called **the equivalent bending moment.**

The equivalent moment is a conventional bending moment, calculated by means of a theory of strength, which equates a compound bending and torsion load, only for the circular or annular section shafts, subjected to bending.

Similarly proceeding with all relations (9.16), the following expressions for the equivalent bending moment result:

I) $M_{ech} == 0{,}5 \cdot (M_i + \sqrt{M_i^2 + M_t^2}),$

II) $M_{ech} = 0{,}35 \cdot M_i + 0{,}65 \cdot \sqrt{M_i + M_t^2},$

III) $M_{ech} = \sqrt{M_i^2 + M_t^2},$

IV) $M_{ech} = \sqrt{M_i^2 + 0{,}65 \cdot M_t^2},$

V) $M_{ech} = \sqrt{M_i^2 + 0{,}75 \cdot M_t^2}.$ **(9.23)**

Using relations (9.23), the value of the equivalent bending moment is obtained. **It is employed in calculating strength as if the shaft were only subjected to bending by a moment having the value M_{ech}.**

Therefore, **the calculation of strength of the circular and annular cross section shafts** subjected to M_i and M_t will be similar to that presented for bending, namely:

a) **verification problems:**

$$\sigma_{ech} = \frac{M_{ech}}{W_z} \leq \sigma_a, \qquad \textbf{(9.24)}$$

b) **maximum allowable load problems:**

$$M_{echcap} = \sigma_a \cdot W_z, \qquad \textbf{(9.25)}$$

c) **dimensioning problems:**

$$d_{nec} = \sqrt[3]{\frac{32M_{ech}}{\pi \cdot \sigma_a}} \quad \text{or} \quad D_{nec} = \sqrt[3]{\frac{32M_{ech}}{(1-k^4) \cdot \pi \cdot \sigma_a}}. \qquad \textbf{(9.26)}$$

Exercise 9.2 Calculate the dimensions of the shaft in figure (9.8, a), made from OL 50 with σ_a = 80 MPa knowing that it has an annular cross section with d = 0.8 D.

Solution: Forces P and Q at the end of the two wheels are reduced in the centers of the respective wheels, resulting in the loading diagram in figure (9.12, b), whereby the shaft is subjected to bending by forces P and Q (the shear load is neglected) and to torsion by moments M_t, $M_{t3} = P \cdot R_1$ and $M_{t4} = Q \cdot R_2$.

The equilibrium equation $M_{tx} = 0$ help us determine load Q:

$$Q = \frac{P \cdot R_1 - M_t}{R_2} = \frac{20 \cdot 0.2 - 2.4}{0.4} = 4kN.$$

The torsion moments are:

$$M_{t1-3} = -M_t = -2.4kNm,$$

$$M_{t3-4} = -M_t + P \cdot R_1 = -2.4 + 20 \cdot 0.2 = 1.6kNm,$$

$$M_{t4-2} = 0,$$

and the diagram of the torsion moments is shown in figure (9.12,c).

The reactions in the bearings are:

$$V_1 = \frac{20 \cdot 1 + 4 \cdot 0.4}{1.2} = 18kN \text{ and}$$

$$V_2 = \frac{20 \cdot 0.2 + 4 \cdot 0.8}{1.2} = 6kN,$$

while the bending moments are: $M_3 = V_1 \cdot 0.2 = 18 \cdot 0.2 = 3.6kNm$ and $M_4 = V_2 \cdot 0.4 = 6 \cdot 0.4 = 2.4kNm$.

The diagram of the bending moments is shown in figure (9.12,d).

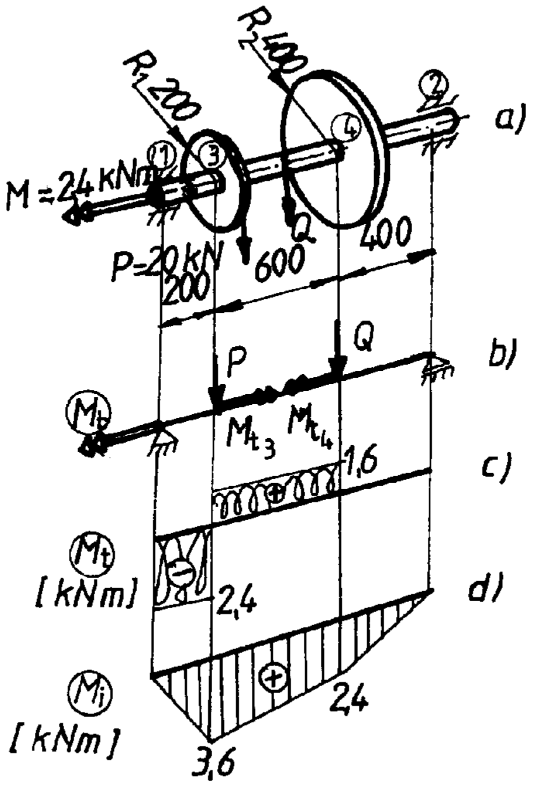


Fig. 9.8

The hazardous section, where the strength is calculated, is section (3) where M_i and M_t have maximum values (in absolute value) and for this section the equivalent moment is:

$$M_{echV} = \sqrt{M_i^2 + 0.75 \cdot M_t^2} = \sqrt{3.6^2 + 0.75 \cdot 2.4^2} = 4.157kNm.$$

The required diameter determined by relation 9.26 for the annular section is:

$$D_{nec} = \sqrt[3]{\frac{32M_{ech}}{\pi \cdot \sigma_a \cdot (1-k^4)}} = \sqrt[3]{\frac{32 \cdot 4.157 \div 10^6}{\pi \cdot 80 \cdot (1-0{,}8^4)}} = 96.42mm.$$

We take: D = 95 mm and d = 76 mm.

Since a lower value than the one calculated was adopted, the verification will be mandatorily done to check to see if it has been exceeded by more than 5%· σ_a.

$$\sigma_{\max} = \frac{32 \cdot M_{ech}}{\pi \cdot D_{efc}^3 \cdot \left(1-k^4\right)} = \frac{32 \cdot 4.157 \cdot 10^6}{\pi \cdot 95^3 \cdot \left(1-0.8^4\right)} = 83.65MPa \lhd 1.05 \cdot \sigma_a = 84MPa. \qquad \textbf{(9.33)}$$

The order of operations in the calculation of strength of arbitrary cross section beams is as follows:

a) The stress diagrams are drawn, the hazardous sections are highlighted (where the stresses reach maximum) and the values of the stresses in each hazardous section are noted. When calculating the maximum allowable load, it is recommended that instead of values to write the expressions of the stresses.

b) The strength calculation required by the respective problem is performed, namely:

- **the beam verification calculation**: consists of calculating and plotting the stress variation diagram for each stress acting upon the hazardous section. For the points of the section with maximum stresses, the equivalent stresses are calculated and compared to the allowable stress;

- **the allowable load**: In this case the loads and the stresses must be expressed according to the unknown load P, then the allowable load P is determined from this condition $\boldsymbol{\sigma}_{\mathbf{max}} \leq \boldsymbol{\sigma}_{\mathbf{a}}$. This calculation is possible only if the expressions of the stress can be expressed according to a single parameter and namely force P.

The dimensioning of the beam subjected to compound load is actually a pre-dimensioning where it is considered:

$$\boldsymbol{\sigma}_{\mathbf{ap}} = \mathbf{(0{,}5\ldots0{,}9)} \cdot \boldsymbol{\sigma}_{\mathbf{a}}\ , \qquad \textbf{(9.34)}$$

and the dimensions of the section are calculated taking into account only the predominant stress. The dimensions are set and then the verification is done by taking into account the stresses caused by all the loads in the hazardous section.

Exercise 9.3 Verify the beam in figure 9.13 knowing that it is made of OL 70 with $\boldsymbol{\sigma_a}$=180 MPa.

Solution: The stress diagrams are shown below the beam and it is observed that the hazardous section is the built-in section (section B).

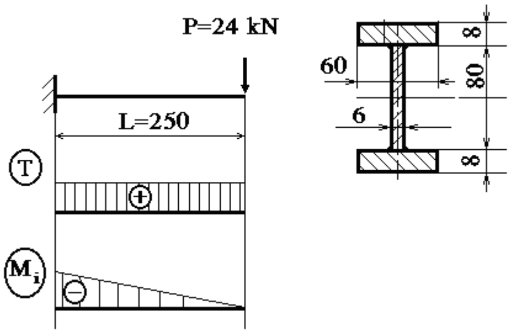


Fig. 9.3

The geometric quantities required are:

$$I_z = (I_{zoi} + y_{0i}^2 \cdot A_i) = \frac{6 \cdot 9.6^3 - 5.4 \cdot 8^3}{12} = 212\, cm^3 ,$$

$$S_1 = S_4 = 0 ,$$

$$S_2 = S_3 = 6 \cdot 0.8 \cdot 4.4 = 21.12\, cm^3 ,$$

$$S_G = S_2 + 4 \cdot 0.6 \cdot 2 = 25.92\, cm^3 ,$$

The stresses corresponding to the loads in the hazardous section are:

- for bending:

$$\sigma_1 = -\sigma_4 = \frac{M_i \cdot y_i}{I_z} = \frac{-6 \cdot 10^6 \cdot (-48)}{212 \cdot 10^4} = 135.9\, MPa ,$$

$$\sigma_2 = -\sigma_3 = \frac{M_i \cdot y_3}{I_t} = \frac{-6 \cdot 10^6 \cdot (-40)}{212 \cdot 10^4} = 113.2\, MPa ,$$

- for shearing:

$$\tau_{xy1} = \tau_{xy4} = 0$$

$$\tau_{xy2t} = \tau_{xy3t} = \frac{T \cdot S_2}{b_{2t} \cdot I_z} = \frac{24 \cdot 10^3 \cdot 21.12 \cdot 10^3}{60 \cdot 212 \cdot 10^4} = 3.985\, MPa ,$$

$$\tau_{xy2i} = \tau_{xy3i} = \frac{T \cdot S_2}{b_{2i} I_z} = \frac{24 x 10^3 x 21.12 x 10^3}{6 x 212 x 10^4} = 39.85 MPa ,$$

$$\tau_G = \frac{T \cdot S_G}{b_G \cdot I_z} = \frac{24 \cdot 10^3 \cdot 25.92 \cdot 10^3}{6 \cdot 212 \cdot 10^4} = 48.91 MPa ,$$

The stress variation diagrams on the hazardous section are shown in figure 9.4.

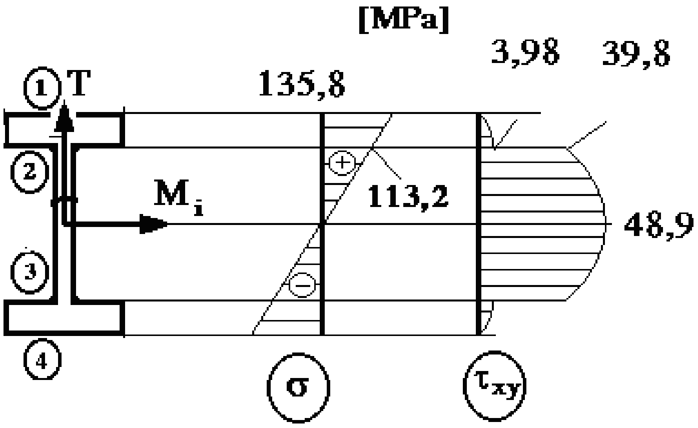


Fig. 9.4

Calculating the equivalent stresses by means of one of the theories of strength (5th theory) and comparing them with the allowable stress, the following are obtained:

$$\sigma_{ech_1} = \sigma_{ech_4} = \sqrt{\sigma_1^2 + 3\cdot\tau_1^2} = \sqrt{135.9^2 + 0} = 135.9\,MPa < \sigma_a\,,$$

$$\sigma_{ech_{2t}} = \sigma_{ech_{3t}} = \sqrt{\sigma_2^2 + 3\cdot\tau_{2t}^2} = \sqrt{113.2^2 + 3\cdot(3.985)^2} = 113.41 MPa < \sigma_a$$

$$\sigma_{ech_{2i}} = \sigma_{ech_{3i}} = \sqrt{\sigma_2^2 + 3\cdot\tau_{2i}^2} = \sqrt{113.2^2 + 3\cdot(39.85)^2} = 151.7\,MPa < \sigma_a$$

$$\sigma_{echG} = \sqrt{3}\cdot\tau_G = \sqrt{3}\cdot(48.91) = 84.71 MPa < \sigma_a$$

The beam endures.

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